Squigonometry
(doing trig on a squircle)

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The point of radian measure (although this probably isn't apparent when we first learn it) is to give a parameterization of the unit circle in terms of arc length. The θ in $\cos(\theta)$ and $\sin(\theta)$ doesn't really represent an *angle*, it is distance (along the curve).
By substituting different curves in place of $x^2 + y^2 = 1$ we get alternate versions of trigonometry.

1. Introduction

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2. Motivation

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3. the Squircle

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4. Squine and Cosquine

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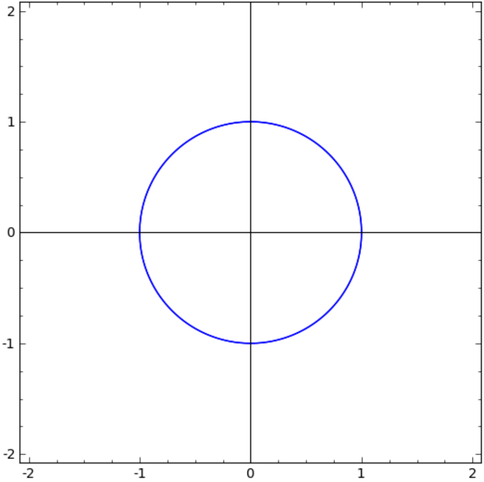
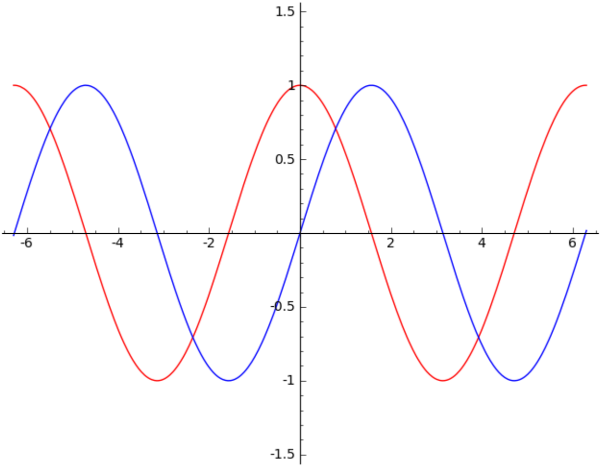
3. That way time t and distance d (along the curve) are equal.
4. $x(t) = \cos(t)$ and $y(t) = \sin(t)$.
5. The equation of the circle produces the fundamental identity
$$\cos^2(t) + \sin^2(t) = 1$$

the geometry of a circle



produces the sine and cosine functions



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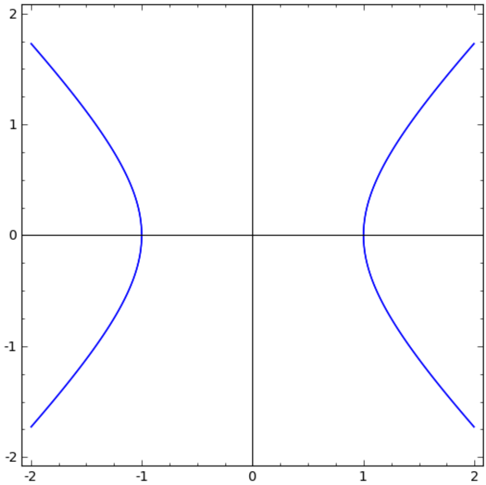
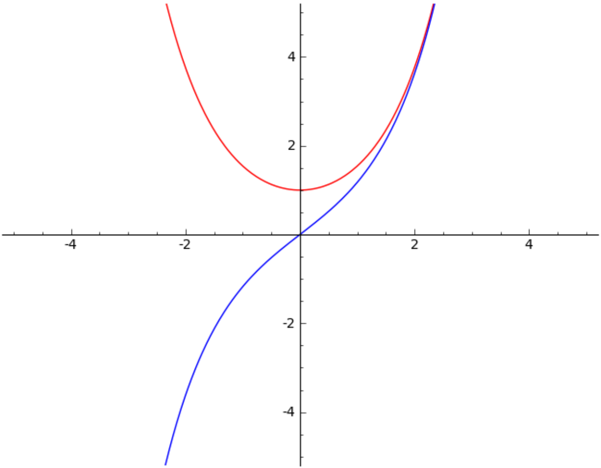
4. Let $t = 0$ correspond to $(1, 0)$.
5. $x(t) = \cosh(t)$ and $y(t) = \sinh(t)$.
6. The equation of the hyperbola forces the these functions to satisfy
$$\cosh^2(t) - \sinh^2(t) = 1$$

the geometry of a hyperbola



produces the hyperbolic sine and cosine functions



The solution to how to go from A to B

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A problem from mechanism design

Finding the optimal motion profile for a transfer-dwell-return-dwell cam, has lead to the discovery of a curious family of functions. Functions that are self-similar to one of their derivatives will be said to satisfy a fractal differential equation.

Automation

- Parts are moved from one station to another where they are incrementally worked on.

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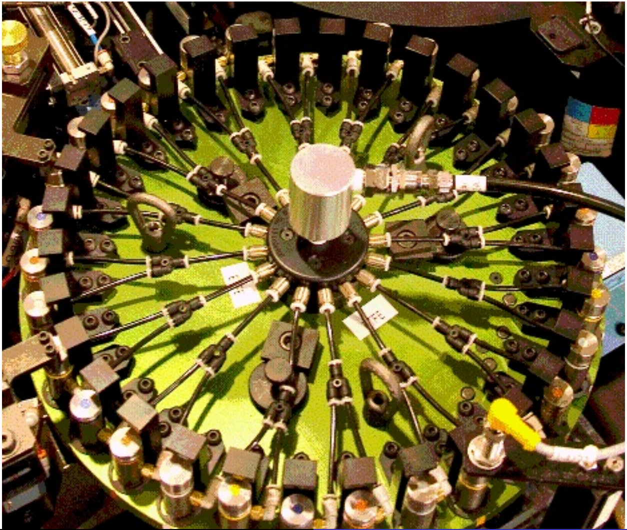
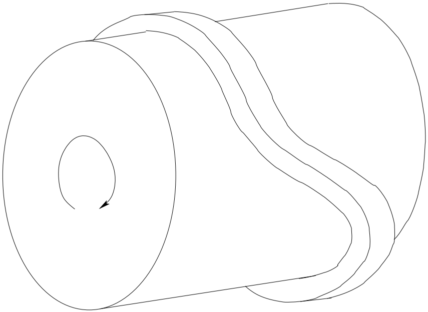
- There are linear and circular transfer systems

a linear turnkey system

a rotary system



barrel cam



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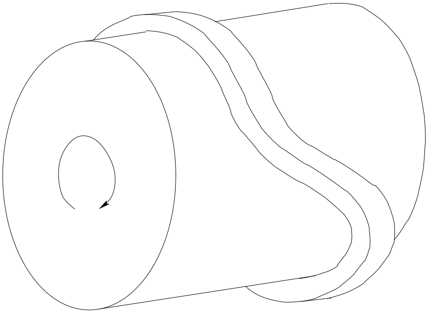
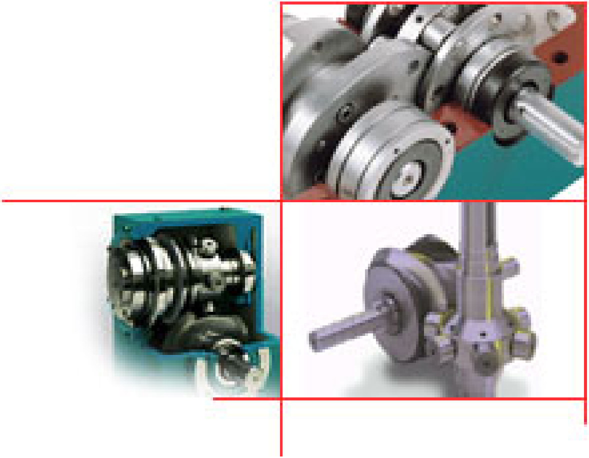


Cam followers are pairs of rollers that are often pre-stressed against the cam rib.

roller gear



the venerable geneva mechanism

http://www.youtube.com/watch?v=mEShmrrdFQw
http://youtu.be/ITQf8JRKndE

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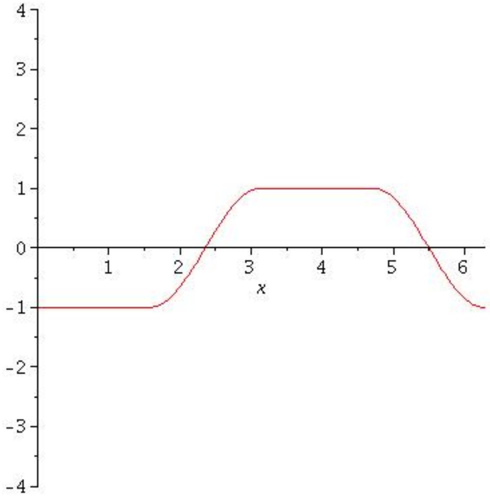
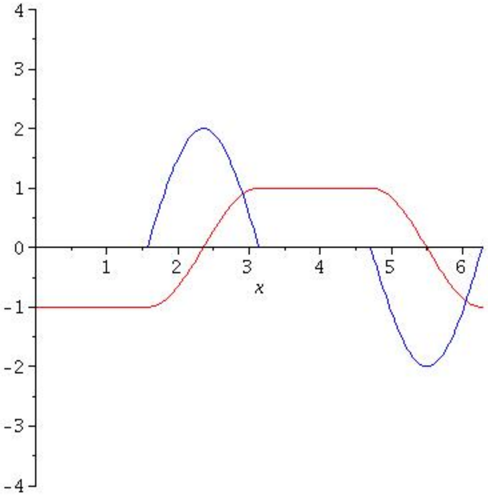
- The two dwells occur at -1 and 1.

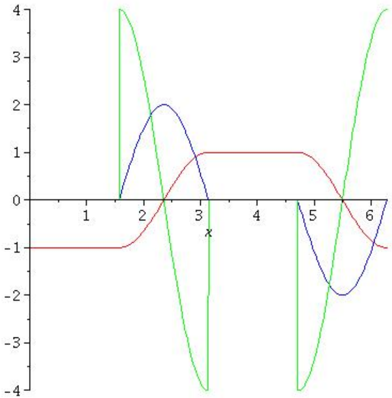
Why not just “patch in” a sine curve?



velocity of piecewise curve using sine



acceleration of piecewise curve using sine



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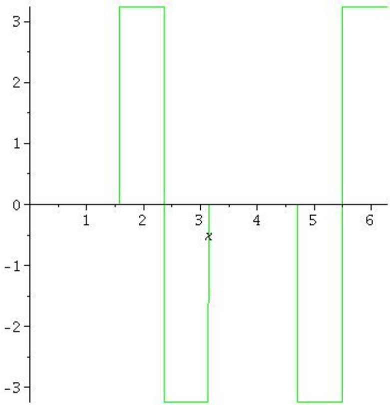
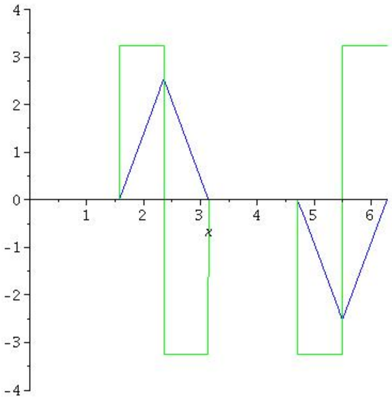
- The motion profile will be piecewise constants and quadratics.

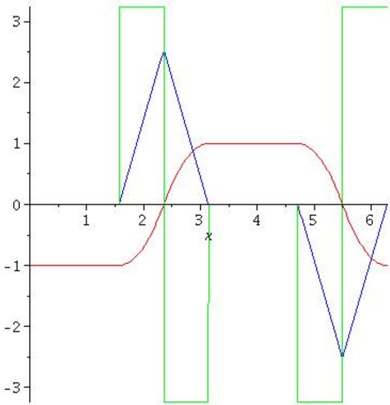
A constant acceleration scheme



A constant acceleration scheme (acc. and vel.)



A constant acceleration scheme (pos., vel. & acc.)



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- Infinitudes in the jerk produce very high transient accelerations. (shocks)

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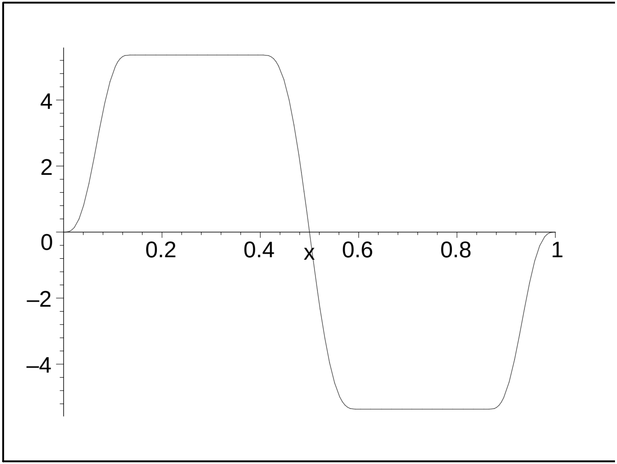
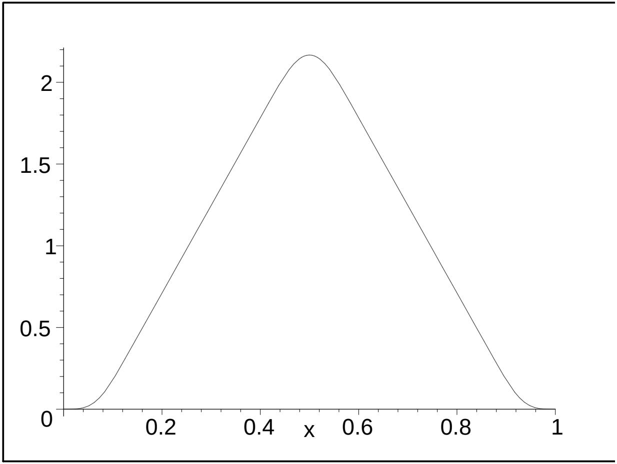
- Self-similarity (in the 2nd derivative)
- The acceleration profile needs to accomplish exactly the same task as the position – transfer from one fixed value to another smoothly.
- The second derivative should be cobbled together out of constants and pieces that look like scaled versions of the original function.

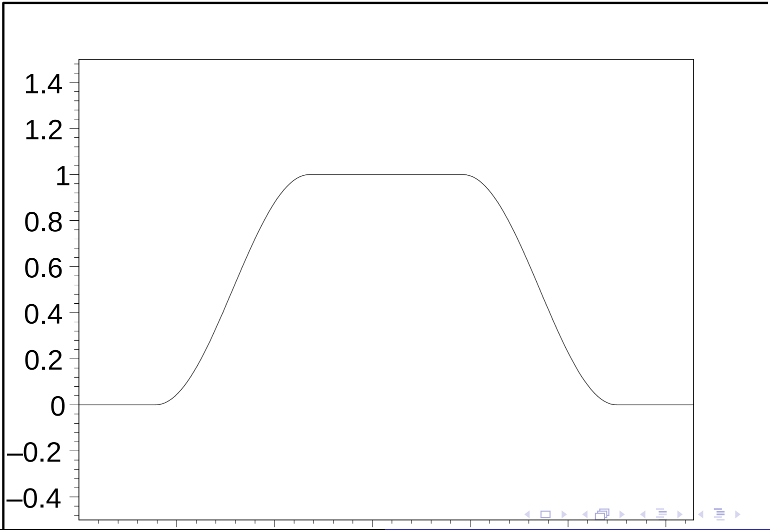
the acceleration profile of a the “fractal cam”



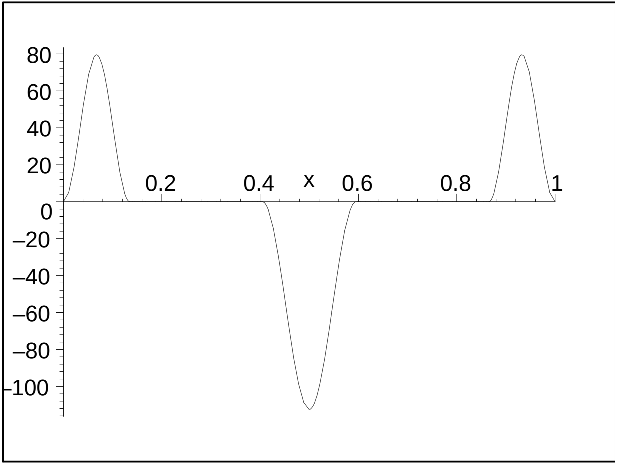
the velocity profile of a the “fractal cam”



the position profile of a the “fractal cam”



the jerk profile of a the "fractal cam"



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- (for instance it is just quadratic in the regions where the acceleration is constant)
- The Taylor series at any point is just a polynomial – it converges to the fractal cam profile only in a small region.
- "Most" Taylor expansions around a point converge on miniscule intervals.

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- Consider the intervals where the profile "dwells."
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- Outside the dwell intervals this Taylor series clearly has no predictive power.

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- Okay, so Taylor wasn't the right way to go, but Fourier ...

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- The coefficients show no discernable pattern.
- The successive Fourier approximations don't converge terribly well to the cam.

question

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- alternate trigonometries from squircles a.k.a. "fat circles" (e.g. $x^4 + y^4 = 1$)