ALL $\mathbb{Z}_4$ CODES OF TYPE II AND LENGTH 16 ARE KNOWN

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Abstract. We show that there are 133 inequivalent Type II codes over $\mathbb{Z}_4$ of length 16. We give the number of each type $4^i \cdot 2^j$, where $2i + j = 16$, a generator matrix for each code, the order of its automorphism group, and its minimum Lee weight. A (partial) symmetrized weight enumerator of each code is given. The highest minimum Lee weight amongst these codes is 8, and there are 5 inequivalent codes attaining this highest Lee weight.

A new computer algorithm to determine the automorphism group of a $\mathbb{Z}_4$ code was used.

1. Introduction

In [2] Conway and Sloane classified the $\mathbb{Z}_4$ self-dual codes of length 8. They were able to do this at length 8 without a mass formula, which was unknown then, and which they stated as an open problem. Recently, Gaborit [3] has found this formula – not only for the number of self-dual $\mathbb{Z}_4$ codes of any length, but also for the Type II codes which exist only at lengths divisible by eight. A self-dual $\mathbb{Z}_4$ code is Type II if the Euclidean weight (1's and 3's count as 1, and 2's count as 4) of every vector is divisible by 8, and if it contains an analogue of the all-ones vector: a vector consisting solely of 1's and 3's.

Using a new computer algorithm that determines the automorphism group of a $\mathbb{Z}_4$ code, we have found 133 inequivalent Type II codes of length 16. By Gaborit's mass formula, we know that this list is complete.

There were 4 Type II codes of length 8, one each of types 4^4, $4^3 2^2$, $4^2 2^4$ and $4 \cdot 2^6$ [2]. The numbers of Type II codes of length 16 are summarized in Table I.

Of these codes, 10 are direct sums of the four Type II codes of length 8, and the rest are indecomposable. For each code we give the order of its automorphism group (which consists of permutations and sign changes) and the code's minimum Lee weight. The highest minimum Lee weight is 8, and there are 5 codes of this weight, one of each of the types $4^5 2^6$, $4^6 2^4$, $4^7 2^2$ and two of type 4^8. We have

Table I

The number of inequivalent self-dual $\mathbb{Z}_4$ codes of Type II by type.

k	1	2	3	4	5	6	7	8
number	1	2	5	13	25	36	34	17

found several codes with the same symmetrized weight enumerator and automorphism group order but which are not equivalent, including the last two mentioned of minimum Lee weight 8.

In order to explain our terminology and methods we give preliminaries and a simpler proof of Gaborit's theorem for the number of $\mathbb{Z}_4$ self-dual and Type II codes of a fixed length n. We also give a recursion for $\tau(n, k)$, the number of doubly-even $[n, k]$ binary codes containing the all-ones vector when $n \equiv 0 \pmod 8$.

There has been much interest in $\mathbb{Z}_4$ codes as they have been shown to be a systematic way of constructing very good binary codes. A more recent interest in $\mathbb{Z}_4$ codes of Type II is their use in building unimodular lattices [1].

We give generator matrices for those codes having minimum Lee weight 8 in Appendix B. Generator matrices for all of the codes are available on the world wide web at `http://math.uic.edu/ fields/z4`.

2. Preliminaries

A code C over $\mathbb{Z}_4$ of length n is an additive subgroup of $\mathbb{Z}_4^n$. The standard inner product is defined on $\mathbb{Z}_4^n$ and self-orthogonal ($C \subset C^\perp$) and self-dual ($C = C^\perp$) codes are defined in the usual way.

Two $\mathbb{Z}_4$ codes are *equivalent* if one can be obtained from the other by permuting coordinates and making sign changes in a subset of the coordinates.

The automorphism group of a length n code over $\mathbb{Z}_4$ is the group of all coordinate permutations and sign changes that fix C set-wise. These groups have natural permutation representations on $2n$ points. There is a normal subgroup consisting of sign changes only, and the quotient group acts naturally on n points.

Any $\mathbb{Z}_4$ code C is equivalent to a code with a generator matrix of the form:

$$G = \begin{bmatrix} I_{k_1} & A & B_1 + 2B_2 \\ 0 & 2I_{k_2} & 2C \end{bmatrix} \tag{1}$$

where the matrices A, B_1, B_2 and C are binary matrices. C contains $4^{k_1}2^{k_2}$ vectors and is said to be of type $4^{k_1}2^{k_2}$. There are two binary codes C_1 and C_2 with generator matrices G_1 and G_2 associated with C in a natural way.

$$G_1 = \left[\begin{array}{ccc} I_{k_1} & A & B_1 \end{array} \right]$$

$$G_2 = \left[\begin{array}{ccc} I_{k_1} & A & B_1 \\ 0 & I_{k_2} & C \end{array} \right]$$

Clearly, $C_1 \subseteq C_2$. If C is self-orthogonal, it is not hard to see that C_1 is doubly-even (hence self-orthogonal) and $C_2 \subseteq C_1^{\perp}$.

Theorem 1. [3] *A $\mathbb{Z}_4$ code C with generator matrix of the form (1) is self-dual iff C_1 is doubly-even, $C_2 = C_1^{\perp}$ and B_2 is chosen so that the rows of G are mutually orthogonal (i.e. the inner-products are 0 in $\mathbb{Z}_4$).*

In this self-dual situation, $k_2 = n - 2k_1$.

Corollary 1. *A $\mathbb{Z}_4$ code C is self-dual iff it has a generator matrix of the form*

$$G = \left[\begin{array}{ccc} D & E & I_k + 2B \\ 0 & 2I_{n-2k} & 2C \end{array} \right]$$

where B, C, D and E are binary matrices,

$$G_1' = \left[\begin{array}{ccc} D & E & I_k \end{array} \right]$$

is a generator matrix for a doubly-even binary code C_1,

$$G_2' = \left[\begin{array}{ccc} D & E & I_k \\ 0 & I_{n-2k} & C \end{array} \right]$$

is a generator matrix for $C_2 = C_1^{\perp}$ and B is chosen in such a way that the first k rows of G are orthogonal (in $\mathbb{Z}_4$).

Proof. If $C_1 = C_2^{\perp}$, the last $n - k_1$ positions of G_2 are information positions. Hence B_1 can be replaced by I_{k_1}. This gives a new generator matrix of C_2 of the following form

$$G_2' = \left[\begin{array}{ccc} D & E & I_k \\ 0 & I_{n-2k} & C \end{array} \right]$$

and the corollary follows from this. $\square$

The Lee weight of a $\mathbb{Z}_4$-vector counts a 1 or 3 component as one, and a 2 component as two. The symmetrized weight enumerator or *swe* of a $\mathbb{Z}_4$-code gives the Lee weights of all its codewords. The Euclidean weight of a vector counts a 1 or 3 component as one and a 2 component as four.

A $\mathbb{Z}_4$-code C is of *Type II* if it is self-dual, contains a vector consisting of 1's and 3's only, and all Euclidean weights in C are divisible by 8. It is known that $\mathbb{Z}_4$ codes of Type II and length n exist iff n is divisible by 8. Further, it is known that a $\mathbb{Z}_4$ code generated by vectors whose Euclidean weights are divisible by 8 and which are mutually orthogonal will only contain vectors of Euclidean weight divisible by 8 [1].

3. Formulas

In an attempt to find a generator matrix for a self-dual $\mathbb{Z}_4$-code we look at the generator matrix in Corollary 1. If we start with the generator matrix of an $[n, k]$ doubly-even code C_1 of the form

$$G_1 = \begin{bmatrix} A & I_k \end{bmatrix}$$

and extend this to a generator matrix

$$G_2 = \begin{bmatrix} G_1 \\ C \end{bmatrix}$$

of $C_1^\perp$, then we can find a generator matrix G (of a self-dual $\mathbb{Z}_4$ code) of the form

$$G = \begin{bmatrix} A & I_k + 2B \\ \multicolumn{2}{c}{2C} \end{bmatrix} \tag{2}$$

for A, B and C binary matrices, provided only that we know how to find $2B$ (we regard $I_k + 2B$ as one matrix where the sum is taken in $\mathbb{Z}_4$).

We illustrate this for a self-dual code of type $4^2 2^4$. There are two doubly-even $[8, 2]$ binary codes one of which contains the all-ones vector and the other does not. Let C_1 be the latter and consider the following G_1.

$$G_1 = \left[\begin{array}{cccccc|cc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

In the $\mathbb{Z}_4$ world we can add 2's (or not) to the entries of the identity matrix on the right, but in order that these new vectors be orthogonal the off-diagonal entries must be different. This generalizes; entries below the diagonal are determined by those above it. This gives the following 2^3 generator matrices of $4^2 2^4$ codes.

$$\left[\begin{array}{cccccc|cc} 1 & 1 & 1 & 0 & 0 & 0 & \pm1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2 & \pm1 \\ & & & & 2C & & & \end{array}\right] \text{ and } \left[\begin{array}{cccccc|cc} 1 & 1 & 1 & 0 & 0 & 0 & \pm1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & \pm1 \\ & & & & 2C & & & \end{array}\right],$$

where

$$2C = \left[\begin{array}{cccccccc} 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \end{array}\right]$$

If we wish to construct a Type II code of type $4^2 2^4$ we must start with an $[8,2]$ doubly-even binary code which contains the all-ones vector. There is exactly one such (up to equivalence).

We see in this situation that there are 2^2 generator matrices possible if we include the all-one vector in our generator matrices.

The possibilities are as follows where $x = \pm1$ and $y = \pm1$.

$$\left[\begin{array}{cccccc|cc} 1 & 1 & 1 & 1 & 1 & 1 & x & -y \\ 1 & 1 & 1 & 0 & 0 & 0 & 2 & y \\ \hline 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \end{array}\right]$$

Note that in this situation the 2 in the lower left corner of the upper-right 2×2 block is determined. This is necessary to ensure that the Euclidean weight of the second basis element be divisible by 8. Furthermore, note that the element in the upper-right corner of this same block is determined by the requirement that these first 2 rows be orthogonal.

So, for Type II codes the relevant generator matrix is

$$G = \left[\begin{array}{ccc} A & & \tilde{I}_k + 2B \\ & 2C & \end{array}\right] \tag{3}$$

where $\tilde{I}_k = \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ 0 & & & \\ \vdots & & I_{k-1} & \\ 0 & & & \end{array}\right]$.

Theorem 2. [3] *There are*

$$\sum_{0 \le k \le \lfloor n/2 \rfloor} \sigma(n,k) 2^{k(k+1)/2}$$

self-dual $\mathbb{Z}_4$ codes, where $\sigma(n,k)$ is the number of doubly-even, binary $[n,k]$ codes.

Further, each term in the sum gives the number of codes of type $4^k 2^{n-2k}$.

Proof. For each binary doubly-even $[n,k]$ code, there are $2^{k(k+1)/2}$ ways of choosing a matrix B such that the generator matrix

$$G = \begin{bmatrix} D & E & I_k + 2B \\ 0 & 2I_{n-2k} & 2C \end{bmatrix}$$

is that of a self-dual code. The entries of B below the diagonal are determined by those above. Summing over all k where $0 \le k \le \lfloor n/2 \rfloor$ gives the result. □

Theorem 3. [3] *Let n be divisible by 8. There are*

$$\sum_{1 \le k \le n/2} \tau(n,k) 2^{1+k(k-1)/2}$$

$\mathbb{Z}_4$ codes of Type II, where $\tau(n,k)$ is the number of doubly-even binary $[n,k]$ codes containing the all-ones vector.

Further, the number of codes of each type are given by the individual summands.

Proof. In the Type II situation we look at a generator matrix of the form in Equation 3 and notice that we can add 2's (or not) freely to the diagonal elements in $\tilde{I}_k$ and to the 0's above the diagonal. However the rest of the matrix is completely determined by these choices. The elements below the diagonal are determined by the choices made above the diagonal and the requirement that the final code be self-orthogonal. The elements in the first column are then determined by the requirement that each row have Euclidean weight zero (mod 8). Finally, the elements in the first row (after the first) must be chosen in such a way that the remaining rows are orthogonal to the first. □

We give a simple recursion for $\tau(n,k)$ when $n \equiv 0 \pmod 8$.

Theorem 4. *When $n \equiv 0 \pmod 8$,*

$$\tau(n,1) = 1$$

and

$$\tau(n,k) = \tau(n,k-1) \cdot \frac{2^{n-2k+1} + 2^{n/2-k} - 1}{2^{k-1} - 1}$$

Proof. The proof is by induction on k.

Clearly $\tau(n,1) = 1$. Suppose that there are $\tau(n,k-1)$ doubly-even $[n,k-1]$ binary codes which contain the all-ones vector. Let C be such a code. We consider the number of doubly-even $[n,k]$ codes C' which contain C.

As $n \equiv 0 \pmod 8$, $C^{\perp}$ is a space of hyperbolic type and so contains $2^{n-k} + 2^{n/2-1}$ vectors whose weights are divisible by 4 [7]. Hence there are $2^{n-k} + 2^{n/2-1} - 2^{k-1}$ vectors which we can adjoin to C to obtain an $[n,k]$ doubly-even code. We divide this by 2^{k-1} (the number of vectors we could adjoin to get the same code C'). Dividing by the $2^{k-1} - 1$ repetitions of the same C' gotten from different C yields the recursion. □

4. The Classification of the Binary Codes

We call a binary code *admissible* if it is doubly-even, and contains the all-ones vector.

We proceeded through the codes of type $4^i 2^j$, $2i + j = 16$ in order of increasing i. We first found all inequivalent, admissible binary codes of length 16 and dimension i using the formula for $\tau(16,i)$ and checking whether we had all the codes by finding the order of the group of each code. As the index of a code's automorphism group in the symmetric group S_{16} gives the number of equivalent codes, we were able to verify that our listing of inequivalent codes was complete. As only self-dual doubly-even codes of length 16 have been classified, we give the classification of all doubly-even binary codes of length 16 and dimension i for $2 \le i \le 7$ in Table II. When $i = 1$, there is clearly one binary code – the length 16 repetition code.

We use the standard notation d_n, e_7 and e_8 for those codes which contain vectors of weight 4. Thus, e_7 is the $[7,3,4]$ Hamming code, e_8 is the $[8,4,4]$ extended Hamming code, and d_n is the code (of even length n) generated by vectors of weight 4 as follows:

$$D_n = \begin{bmatrix} 1 & 1 & 1 & 1 & & & & & & & & & & & \\ & & 1 & 1 & 1 & 1 & & & & & & & & & \\ & & & & 1 & 1 & 1 & 1 & & & & & & & \\ & & & & & & & & \ddots & & & & & & \\ & & & & & & & & & 1 & 1 & 1 & 1 & & & \\ & & & & & & & & & & & 1 & 1 & 1 & 1 & & \\ & & & & & & & & & & & & & 1 & 1 & 1 & 1 \end{bmatrix}$$

Our labels describe all the vectors of weight four in an admissible code of a given dimension. A '+' sign in a label indicates a direct sum of weight 4 component sub-codes. As these components may need to be 'glued' together, the resulting code may not be a direct sum. The same labels may be used in different dimensions, so a numeral is prefixed to indicate the dimension (the binary code's dimension gives the exponent of 4 in the type of the related $\mathbb{Z}_4$ code). Thus 4_d4+d6 is an admissible code (not a direct sum) of dimension 4 with the following generator matrix

$$\text{4\_d4+d6} = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0 \end{bmatrix}.$$

It generates a $\mathbb{Z}_4$ code of type 4^4 which we also label **4_d4+d6**. We use bold face type to distinguish the labels of $\mathbb{Z}_4$ codes from those of binary codes.

There is no code with the label *X*_d4+d8 as such a code would contain an additional weight four vector. (Recall that the all-ones vector is always in the code.)

The label 5_d4+d6 refers to the code (of dimension 5) with generator matrix

$$\text{5\_d4+d6} = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0 \\ 1&0&1&0&1&0&1&0&1&0&1&1&1&0&0&0 \end{bmatrix}.$$

Note that the final row is a weight 8 'glue' vector, it does not introduce any new weight 4 vectors into the code. Again, this code is not a direct sum.

There are 40 inequivalent admissible codes of length 16. The unique code of dimension 1 is the length 16 repetition code; at the other end of the spectrum, there are two codes (necessarily of dimension 8): d16 and 2e8. These have previously been classified [6].

These codes are completely described by labels which indicate the weight four vectors in them. A '+' sign in a label does not necessarily indicate a *direct* sum, as there may be 'glue' vectors of weight 8. In Table II we give the weight distributions and the order of the automorphism groups of all these codes. Note that the entire weight distribution is determined by the number of vectors of weights 4 and 8 as the all-one vector is in the code. The generator matrices of these codes are not difficult to find from their labels.

TABLE II

Inequivalent admissible codes of length 16 and dimensions 2-7.

Label	Group Order	u_4	u_8
2_d4	$2^{13}3^6 5^2 7\cdot 11$	1	1
2_f2	$2^{15}3^4 5^2 7^2$	0	2
3_d4	$2^{12}3^4 5^2$	1	4
3_2d4	$2^{14}3^4 5\cdot 7$	2	2
3_d6	$2^{12}3^5 5^2 7$	3	0
3_f3	$2^{15}3^5$	0	6
4_d4	$2^9 3^5$	1	12
4_2d4	$2^{14}3^2$	2	10
4_d4+d6	$2^{11}3^4 5$	4	6
4_4d4	$2^{15}3^5$	4	6
4_d6	$2^{10}3^3 5^2$	3	8
4_d8	$2^{14}3^3 5\cdot 7$	6	2
4_e7	$2^{10}3^5 5\cdot 7^2$	7	0
4_f4	$2^{14}3\cdot 7$	0	14
5_2d4	$2^{12}\cdot 3$	2	26
5_d4+d6	$2^9 3^3$	4	22
5_d4+e7	$2^9 3^3 5\cdot 7$	8	14
5_2d4+d8	$2^{14}3^3$	8	14
5_4d4	$2^{15}3$	4	22
5_2d6	$2^{12}3^3$	6	18
5_d8	$2^{13}3^3$	6	18
5_d10	$2^{12}3^3 5^2$	10	10
5_e8	$2^{13}3^3 5\cdot 7^2$	14	2
5_f5	$2^{10}3^2 5\cdot 7$	0	30
6_d4+d12	$2^{13}3^3 5$	16	30
6_2d4+d8	$2^{13}3$	8	46
6_2d4+e8	$2^{13}3^3 7$	16	30
6_4d4	$2^{12}3^2$	4	54
6_d6+e7	$2^8 3^3 7$	10	42
6_2d6	$2^{10}3^2$	6	50
6_2d8	$2^{15}3^2$	12	38
6_d10	$2^{10}3^3 5$	10	42
7_d4+d12	$2^{12}3^2 5$	16	94
7_2d8	$2^{13}3^2$	12	102
7_d8+e8	$2^{13}3^2 7$	20	86
7_d16	$2^{15}3^2 5\cdot 7$	28	70
7_2e7	$2^8 3^2 7^2$	14	98

Using our notation, the four $\mathbb{Z}_4$ Type II codes of length 8 found in [2], would have the labels: **1_f1**, **2_2d4**, **3_d8** and **4_e8**.

5. Our Methods

This work relies in an essential way on our ability to compute the order of the automorphism group of a $\mathbb{Z}_4$ code. The automorphism groups of the codes were computed using an algorithm employing the partition backtrack technique developed in [4] and [5]. This techique involves backtrack search in a tree whose nodes are labelled by nested sequences of ordered partitions; a detailed description of the technique appears in [4], and a somewhat briefer description, incorporating a number of small improvements, is given in [5]. An isomorphism algorithm for codes over $\mathbb{Z}_4$, based on the same general technique, was developed and was used in the classification process. However, the final result does not depend on the validity of the isomorphism algorithm, as any two distinct codes (see Appendix A) can be distinguished by their symmetrized weight enumerators (determined by the coefficients shown) and by the orders of their automorphism and sign change groups.

After all of the admissible codes of a fixed dimension were found, we constructed the generator matrices for some of the $\mathbb{Z}_4$ codes to which they lead. We tried to eliminate equivalent codes before running the automorphism group program. The $\mathbb{Z}_4$ codes retained the same labels as their binary progenitors; however there may be more than one $\mathbb{Z}_4$ code with the same label. We distinguish these by appending a,b,c, ... (as necessary) to their labels. For example, for $\mathbb{Z}_4$ codes of type $4^5 2^6$ and label **5_d4+d6**, we considered generator matrices of the form

$$\mathbf{5_d4{+}d6}x = \left[\begin{array}{ccccccccccc|ccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 1 & a & b & c \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \star & a & 1 & d & e \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \star & b & \tilde{d} & 1 & f \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \star & \tilde{c} & \tilde{e} & \tilde{f} & 1 \\
\hline
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0
\end{array}\right]$$

The six positions above the diagonal in the upper-right block (indicated by the variables $a, b, \ldots f$) are considered free, that is, one can put either a zero or a two in those positions, and the positions below the diagonal are determined by those choices. A tilde on a variable name appearing below the diagonal means that this position must have the opposite value as the corresponding variable above the diagonal. The positions marked with stars are also determined, they must be chosen so that the row has Euclidean weight zero (mod 8). Finally, note that in the proof of Gaborit's formula for the total number of Type II codes, the positions *on* the diagonal are taken to be free, i.e. either 1 or 3. We do not consider these positions free – a sign change in any of the last 5 coordinates would send us to an equivalent code in which the opposite choice had been made on the diagonal. So, we see that there are at most 2^6 inequivalent codes with the label **5_d4+d6**x. There are a variety of different methods for eliminating equivalent codes, some of which use symmetries. (In the above example, rows 3 and 4 can be interchanged together with the appropriate column interchanges.) Also elements in the dual of C_1 can be added to some rows. Some of this pruning of the list of candidate codes was accomplished by hand. The final list of codes were distinguished by their symmetrized weight enumerators, their automorphism group orders, the size of the normal subgroup consisting of sign changes, and the orbit lengths of the quotient group which acts on 16 unsigned coordinates.

There are several cases where two codes have the same symmetrized weight enumerators and automorphism group orders but are not equivalent. One instance of this is **8_d16g** and **8_2e8j**, both of minimum Lee weight 8. The orders of their sign change groups are different. In addition, they cannot be equivalent as they have different labels.

The order of the group of each $\mathbb{Z}_4$ code C yields the number of codes equivalent to C. We knew that we were finished when the total number of codes of a given type was equal to the number found in Theorem 3.

There are five admissible codes which do not contain any weight 4 vectors. These are denoted by labels: 1_f1, 2_f2, 3_f3, 4_f4 and 5_f5. One of these (5_f5) is actually $R(1,4)$, 4_f4 is a four-dimensional sub-code of $R(1,4)$. Each of these lead to $\mathbb{Z}_4$ codes of types $4^5 2^6$ and $4^4 2^8$ respectively, that have minimum Lee weight 8. The Gray image of **5_f5** is the doubly-even linear code $R(2,5)$.

The others are also sub-codes of $R(1,4)$ with dimensions 1,2 and 3. They give rise to $\mathbb{Z}_4$ codes of types $4^1 2^{14}$, $4^2 2^{12}$ and $4^3 2^{10}$, these codes have minimum Lee weight 4.

For the codes having minimum Lee weight 8, we give their complete symmetrized weight enumerators in Table III. There is a one-to-one correspondence between certain pairs of vectors that are translates of one another by the all 2's vector. Because of this, certain monomials in the symmetrized weight enumerator of a code have equal coefficients. For example, the coefficients of b^4c^1 and b^4c^{11} are equal in the symmetrized weight enumerator of any Type II code (indeed, this is true for all self-dual $\mathbb{Z}_4$ codes). In Table III, the information for like coefficients is combined.

6. Explanation of the Appendices

In Appendix A, we give a summary of all of the inequivalent Type II codes of length 16. Direct sums are marked with an asterisk. A length 16 Type II code which is a direct sum must be a direct sum of length 8 Type II codes. For example, the direct sum of the length 8 codes **1_f1** and **2_2d4** gives the length 16 code labeled **3_2d4b**. Even though there are 10 codes labeled **8_2e8**x ($x = a, b, \ldots, j$) only one, **8_2e8d** is **4_e8** $\oplus$ **4_e8**.

For each code, we give the order of its automorphism group, the group's orbit lengths, the minimum Lee weight, and some of the coefficients of the symmetrized weight enumerator.

Table III

Symmetrized weight enumerators for codes with minimum Lee weight 8.

label	swe coefficients										
	1 c^{16}	c^4 c^{12}	c^6 c^{10}	c^8	b^4c^3 b^4c^9	b^4c^5 b^4c^7	b^8 b^8c^8	b^8c^2 b^8c^6	b^8c^4	$b^{12}c$ $b^{12}c^3$	b^{16}
5_f5	1	140	448	870	0	0	480	13440	33600	0	2048
6_4d4c	1	76	192	486	512	1536	480	11904	30528	2048	1024
7_2d8g	1	44	64	294	768	2304	480	11136	28992	3072	512
8_d16g	1	28	0	198	896	2688	480	10752	28224	3584	256
8_2e8j	1	28	0	198	896	2688	480	10752	28224	3584	256

Recall that the automorphism group A has a normal subgroup S which consists of sign changes only. The quotient group A/S acts as a permutation group on the 16 unsigned coordinates. In the appendix, we write the automorphism group's order as $(|S|)\cdot|A/S|$. Informally, we have factored the group into pure 'sign change', and pure 'permutation' components. The orbit lengths in Appendix A are the lengths of the orbits of A/S acting on the 16 unsigned coordinate positions.

The symmetrized weight enumerator of a $\mathbb{Z}_4$ code is a polynomial in two variables b and c such that the coefficient of b^ic^j is the number of vectors in the code having exactly i 1's and 3's and j 2's. In Appendix A we give only the coefficients of c^2, b^4c, b^8 and c^4. These are the only terms (other than the leading 1) which correspond to vectors having Lee weight 8 or less. These terms are sufficient to distinguish the symmetrized weight enumerators of all of the codes of Type II and length 16.

In Appendix B, we give generator matrices for all of the inequivalent codes we have determined that have minumum Lee weight 8.

7. Acknowledgments

We thank Phillipe Gaborit for making a preprint of [3] available to us. We found GAP (Groups, Algorithms and Programming) to be an invaluable tool, and would like to thank its authors.

Appendix A: Summary of the $\mathbb{Z}_4$ Type II Codes

All inequivalent $\mathbb{Z}_4$ Type II codes of length 16 are listed below. Direct sums are marked with an asterisk.

	label	automorphism group order	orbit lengths	min Lee wt	swe coefficients (partial)			
					c^2	b^4c	b^8	c^4
	1_f1	$(2^{15})2^{15}3^6 5^3 7^2 11\cdot 13$	16	4	120	0	0	1820
	2_d4	$(2^{14})2^{13}3^6 5^2 7\cdot 11$	4,12	4	72	96	0	892
*	**2_f2**	$(2^{14})2^{15}3^4 5^2 7^2$	16	4	56	0	256	924
	3_d4	$(2^{12})2^{12}3^4 5^2$	4,12	4	32	48	256	460
	3_2d4a	$(2^{13})2^{14}3^4 5\cdot 7$	8,8	4	40	128	64	444
*	**3_2d4b**	$(2^{13})2^{14}3^4 5\cdot 7$	8,8	4	40	64	192	444
	3_d6	$(2^{12})2^{12}3^5 5^2 7$	6,10	4	48	144	0	428
	3_f3	$(2^{12})2^{15}3^5$	16	4	24	0	384	476
	4_d4	$(2^9)2^9 3^5$	4,12	4	12	24	384	244
	4_d4+d6a	$(2^{11})2^{11}3^4 5$	4,6,6	4	24	144	96	220
	4_d4+d6b	$(2^{11})2^{11}3^3 5$	4,4,2,6	4	24	80	224	220
	4_2d4a	$(2^{10})2^{14}3^2$	8,8	4	16	64	288	236
	4_2d4b	$(2^{10})2^{14}3^2$	8,8	4	16	32	352	236
	4_4d4a	$(2^{12})2^{13}3^5$	12,4	4	24	96	192	220
*	**4_4d4b**	$(2^{12})2^{15}3^4$	16	4	24	128	128	220
	4_4d4c	$(2^{12})2^{15}3^5$	16	4	24	0	384	220
	4_d6	$(2^9)2^{10}3^3 5^2$	6,10	4	20	72	256	228
	4_d8a	$(2^{11})2^{14}3^2 5\cdot 7$	8,8	4	32	160	32	204
*	**4_d8b**	$(2^{11})2^{14}3^3 5\cdot 7$	8,8	4	32	96	160	204
	4_e7	$(2^9)2^{10}3^5 5\cdot 7^2$	7,9	4	36	168	0	196
	4_f4	$(2^9)2^{14}3\cdot 7$	16	4	8	0	448	252
	5_d4+d6a	$(2^7)2^9 3^3$	4,6,6	4	8	72	304	124
	5_d4+d6b	$(2^7)2^9 3^2$	4,4,2,6	4	8	40	368	124
	5_d4+e7a	$(2^8)2^9 3^3 5\cdot 7$	7,4,5	4	16	152	112	108
	5_d4+e7b	$(2^8)2^9 3^3 5$	6,4,5,1	4	16	88	240	108
	5_2d4a	$(2^6)2^{12}3$	8,8	4	4	32	400	132
	5_2d4b	$(2^6)2^{12}3$	8,8	4	4	16	432	132
	5_2d4+d8a	$(2^{10})2^{13}3^2$	8,4,4	4	16	128	160	108

	label	automorphism group order	orbit lengths	min Lee wt	swe coefficients (partial) c^2	b^4c	b^8	c^4
	5_2d4+d8b	$(2^{10})2^{12}3^2$	4,4,8	4	16	112	192	108
	5_2d4+d8c	$(2^{10})2^{13}3^2$	8,8	4	16	64	288	108
*	**5_2d4+d8d**	$(2^{10})2^{14}3^3$	8,8	4	16	160	96	108
	5_2d4+d8e	$(2^{10})2^{12}3^3$	6,2,8	4	16	80	256	108
	5_2d4+d8f	$(2^{10})2^{14}3^2$	8,8	4	16	32	352	108
	5_4d4a	$(2^8)2^{13}3$	12,4	4	8	48	352	124
	5_4d4b	$(2^8)2^{15}$	16	4	8	64	320	124
	5_4d4c	$(2^8)2^{15}3$	16	4	8	0	448	124
	5_2d6a	$(2^9)2^{12}3^3$	12,4	4	12	144	144	116
	5_2d6b	$(2^9)2^{12}3$	8,4,4	4	12	80	272	116
	5_2d6c	$(2^9)2^{11}3^2$	12,4	4	12	48	336	116
	5_d8a	$(2^7)2^{13}3^2$	8,8	4	12	80	272	116
	5_d8b	$(2^7)2^{13}3^3$	8,8	4	12	48	336	116
	5_d10a	$(2^{10})2^{10}3^25^2$	10,6	4	20	160	80	100
	5_d10b	$(2^{10})2^{10}3^35$	4,6,6	4	20	96	208	100
	5_e8a	$(2^8)2^{13}3^35\cdot 7$	8,8	4	28	176	16	84
*	**5_e8b**	$(2^8)2^{13}3^35\cdot 7^2$	8,8	4	28	112	144	84
	5_f5	$(2^5)2^{10}3^25\cdot 7$	16	8	0	0	480	140
	6_d4+d12a	$(2^9)2^{11}3^2$	12,4	4	12	144	144	52
	6_d4+d12b	$(2^9)2^{10}3^2$	4,2,6,4	4	12	128	176	52
	6_d4+d12c	$(2^9)2^{11}3$	8,4,4	4	12	80	272	52
	6_d4+d12d	$(2^9)2^{10}3\cdot 5$	10,2,4	4	12	96	240	52
	6_d4+d12e	$(2^9)2^{11}3^3$	12,4	4	12	48	336	52
	6_d4+d12f	$(2^9)2^{11}3^2$	12,4	4	12	48	336	52
	6_2d4+d8a	$(2^5)2^{12}$	8,4,4	4	4	64	336	68
	6_2d4+d8b	$(2^5)2^{11}$	4,4,8	4	4	56	352	68
	6_2d4+d8c	$(2^5)2^{12}$	8,8	4	4	32	400	68
	6_2d4+d8d	$(2^5)2^{13}3$	8,8	4	4	80	304	68
	6_2d4+d8e	$(2^5)2^{11}3$	6,2,8	4	4	40	384	68
	6_2d4+d8f	$(2^5)2^{13}$	8,8	4	4	16	432	68
	6_2d4+e8a	$(2^7)2^{12}3^3$	8,4,4	4	12	144	144	52
	6_2d4+e8b	$(2^7)2^{10}3^3$	6,2,8	4	12	104	224	52
	6_2d4+e8c	$(2^7)2^{12}3^2$	8,8	4	12	80	272	52
*	**6_2d4+e8d**	$(2^7)2^{13}3^37$	8,8	4	12	176	80	52
	6_2d4+e8e	$(2^7)2^{10}3^37$	7,8,1	4	12	72	288	52

	label	automorphism group order	orbit lengths	min Lee wt	swe coefficients (partial) c^2	b^4c	b^8	c^4
	6_2d4+e8f	$(2^7)2^{13}3^3$	8,8	4	12	48	336	52
	6_4d4a	$(2^3)2^{10}3^2$	12,4	6	0	24	432	76
	6_4d4b	$(2^3)2^{12}3$	16	6	0	32	416	76
	6_4d4c	$(2^3)2^{12}3^2$	16	8	0	0	480	76
	6_d6+e7a	$(2^6)2^8 3^3 7$	6,7,3	4	6	144	168	64
	6_d6+e7b	$(2^6)2^8 3^2$	4,2,6,3,1	4	6	80	296	64
	6_d6+e7c	$(2^6)2^7 3^2$	6,4,3,3	4	6	48	360	64
	6_2d6a	$(2^4)2^{10}3^2$	12,4	4	2	72	328	72
	6_2d6b	$(2^4)2^{10}$	8,4,4	4	2	40	392	72
	6_2d6c	$(2^4)2^9 3$	12,4	4	2	24	424	72
	6_2d8a	$(2^8)2^{12}3$	8,6,2	4	8	112	224	60
	6_2d8b	$(2^8)2^{12}$	8,8	4	8	96	256	60
	6_2d8c	$(2^8)2^{12}$	8,4,4	4	8	48	352	60
	6_2d8d	$(2^8)2^{12}$	16	4	8	64	320	60
$*$	**6_2d8e**	$(2^8)2^{15}3^2$	16	4	8	192	64	60
	6_2d8f	$(2^8)2^{15}$	16	4	8	64	320	60
	6_2d8g	$(2^8)2^{14}3$	16	4	8	0	448	60
	6_d10a	$(2^5)2^8 3^2 5$	10,6	4	6	80	296	64
	6_d10b	$(2^5)2^8 3^3$	4,6,6	4	6	48	360	64
	7_d4+d12a	$(2^3)2^{10}3$	12,4	4	2	72	328	40
	7_d4+d12b	$(2^3)2^9 3$	4,2,6,4	4	2	64	344	40
	7_d4+d12c	$(2^3)2^{10}$	8,4,4	4	2	40	392	40
	7_d4+d12d	$(2^3)2^9 5$	10,2,4	4	2	48	376	40
	7_d4+d12e	$(2^3)2^{10}3^2$	12,4	4	2	24	424	40
	7_d4+d12f	$(2^3)2^{10}3$	12,4	4	2	24	424	40
	7_d8+e8a	$(2^5)2^9 3^2$	8,6,2	4	4	128	208	36
	7_d8+e8b	$(2^5)2^9 3$	6,2,4,4	4	4	88	288	36
	7_d8+e8c	$(2^5)2^{11}$	8,4,4	4	4	64	336	36
	7_d8+e8d	$(2^5)2^9$	4,4,8	4	4	56	352	36
	7_d8+e8e	$(2^5)2^{10}3$	8,8	4	4	32	400	36
$*$	**7_d8+e8f**	$(2^5)2^{13}3^2 7$	8,8	4	4	208	48	36
	7_d8+e8g	$(2^5)2^{10}3 \cdot 7$	7,8,1	4	4	104	256	36
	7_d8+e8h	$(2^5)2^{13}3$	8,8	4	4	80	304	36
	7_d8+e8i	$(2^5)2^{10}3$	6,8,2	4	4	40	384	36
	7_d8+e8j	$(2^5)2^{12}3$	8,8	4	4	16	432	36

	label	automorphism group order	orbit lengths	min Lee wt	swe coefficients (partial) c^2	b^4c	b^8	c^4
	7_2d8a	$(2^2)2^{10}3$	8,6,2	6	0	56	368	44
	7_2d8b	$(2^2)2^{10}$	8,8	6	0	48	384	44
	7_2d8c	$(2^2)2^{10}$	8,4,4	6	0	24	432	44
	7_2d8d	$(2^2)2^{10}$	16	6	0	32	416	44
	7_2d8e	$(2^2)2^{13}3^2$	16	6	0	96	288	44
	7_2d8f	$(2^2)2^{13}$	16	6	0	32	416	44
	7_2d8g	$(2^2)2^{12}3$	16	8	0	0	480	44
	7_d16a	$(2^8)2^97$	14,2	4	8	112	224	28
	7_d16b	$(2^8)2^{10}3^2$	4,12	4	8	144	160	28
	7_d16c	$(2^8)2^{11}$	8,8	4	8	96	256	28
	7_d16d	$(2^8)2^{10}3$	12,4	4	8	48	352	28
	7_d16e	$(2^8)2^93\cdot5$	10,6	4	8	64	320	28
	7_d16f	$(2^8)2^{12}$	16	4	8	64	320	28
	7_d16g	$(2^8)2^{12}3\cdot7$	16	4	8	0	448	28
	7_2e7a	$(2^3)2^83^27^2$	14,2	4	1	140	196	42
	7_2e7b	$(2^3)2^83^2$	12,2,2	4	1	76	324	42
	7_2e7c	$(2^3)2^73$	6,8,2	4	1	44	388	42
	7_2e7d	$(2^3)2^53\cdot7$	14,2	4	1	28	420	42
	8_d16a	$(2)2^87$	14,2	6	0	56	368	28
	8_d16b	$(2)2^{10}$	8,8	6	0	48	384	28
	8_d16c	$(2)2^93^2$	12,4	6	0	72	336	28
	8_d16d	$(2)2^93$	12,4	6	0	24	432	28
	8_d16e	$(2)2^83\cdot5$	10,6	6	0	32	416	28
	8_d16f	$(2)2^{11}$	16	6	0	32	416	28
	8_d16g	$(2)2^{11}3\cdot7$	16	8	0	0	480	28
	8_2e8a	$(2^2)2^83^2$	12,4	6	0	80	320	28
	8_2e8b	$(2^2)2^8$	8,8	6	0	48	384	28
	8_2e8c	$(2^2)2^73$	16	6	0	32	416	28
$*$	**8_2e8d**	$(2^2)2^{13}3^27^2$	16	6	0	224	32	28
	8_2e8e	$(2^2)2^{13}3^2$	16	6	0	96	288	28
	8_2e8f	$(2^2)2^93^27$	7,1,8	6	0	120	240	28
	8_2e8g	$(2^2)2^93$	6,2,8	6	0	56	368	28
	8_2e8h	$(2^2)2^{12}3$	16	6	0	32	416	28
	8_2e8i	$(2^2)2^83$	4,4,8	6	0	24	432	28
	8_2e8j	$(2^2)2^{10}3\cdot7$	16	8	0	0	480	28

Appendix B: Generator Matrices

$$\mathbf{5_f5} = \left[\begin{array}{ccccccccccc|ccccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&0&1&1&1&1&1&1&0&0&0&0&1&0&0&0\\
1&1&0&1&0&0&1&1&1&1&0&0&0&1&0&0\\
1&1&1&0&1&0&1&0&1&0&1&0&0&0&1&0\\
0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&1\\
\hline
0&0&0&0&0&2&0&0&0&0&0&2&2&0&0&2\\
0&0&0&0&0&0&2&0&0&0&0&2&2&2&2&2\\
0&0&0&0&0&0&0&2&0&0&0&0&2&2&0&2\\
0&0&0&0&0&0&0&0&2&0&0&0&0&2&2&2\\
0&0&0&0&0&0&0&0&0&2&0&2&0&2&0&2\\
0&0&0&0&0&0&0&0&0&0&2&2&0&0&2&2
\end{array}\right]$$

$$\mathbf{6_4d4c} = \left[\begin{array}{cccccccccc|cccccc}
1&1&1&1&1&1&1&1&1&1&1&3&3&3&1&1\\
1&1&1&0&0&0&0&0&0&0&2&1&2&2&2&2\\
0&0&0&1&1&1&0&0&0&0&2&2&1&2&2&2\\
0&0&0&0&0&0&1&1&1&0&2&2&2&1&2&2\\
1&1&0&1&1&0&1&1&0&1&0&0&0&0&1&0\\
1&0&1&1&0&1&1&0&1&1&0&0&0&0&0&1\\
\hline
0&0&0&0&2&0&0&0&0&0&2&0&2&0&2&0\\
0&0&0&0&0&2&0&0&0&0&2&0&2&0&0&2\\
0&0&0&0&0&0&0&2&0&0&2&0&0&2&2&0\\
0&0&0&0&0&0&0&0&2&0&2&0&0&2&0&2
\end{array}\right]$$

$$\mathbf{7_2d8g} = \left[\begin{array}{ccccccccc|ccccccc}
1&1&1&1&1&1&1&1&1&1&3&3&3&3&3&1\\
1&1&1&0&0&0&0&0&0&2&1&0&2&0&2&0\\
1&1&0&1&0&0&0&0&0&2&2&1&0&2&0&0\\
1&1&0&0&1&0&0&0&0&0&0&2&1&2&2&0\\
0&0&0&0&0&1&1&1&0&2&0&2&2&1&0&0\\
0&0&0&0&0&1&1&0&1&0&2&0&2&2&1&0\\
1&0&1&1&1&1&0&1&1&2&2&2&2&2&2&1\\
\hline
0&0&0&0&2&0&0&0&0&2&0&0&2&0&0&2\\
0&0&0&0&0&0&2&0&0&2&0&0&0&2&2&0
\end{array}\right]$$

$$\mathbf{8_d16g} = \left[\begin{array}{cccccccc|cccccccc}
1&1&1&1&1&1&1&1&1&3&3&3&3&3&3&1\\
1&1&1&0&0&0&0&0&2&1&0&0&0&2&2&0\\
1&1&0&1&0&0&0&0&2&2&1&0&2&0&0&0\\
1&1&0&0&1&0&0&0&0&2&2&1&0&0&2&0\\
1&1&0&0&0&1&0&0&0&2&0&2&1&2&0&0\\
1&1&0&0&0&0&1&0&2&0&2&2&0&1&0&0\\
1&1&0&0&0&0&0&1&0&0&2&0&2&2&1&0\\
1&0&1&1&1&1&1&1&2&2&2&2&2&2&2&1
\end{array}\right]$$

$$\mathbf{8_2e8j} = \left[\begin{array}{cccccccc|cccccccc}
1&1&1&1&1&1&1&1&1&3&3&3&3&3&3&3\\
0&1&1&1&0&0&0&0&2&1&0&0&2&0&0&2\\
1&0&1&1&0&0&0&0&0&2&1&0&0&0&2&2\\
1&1&0&1&0&0&0&0&2&2&2&1&2&2&0&0\\
1&1&1&0&0&0&0&0&0&0&2&0&1&2&2&0\\
0&0&0&0&1&1&1&0&2&0&0&2&2&1&0&0\\
0&0&0&0&1&1&0&1&0&0&2&0&2&2&1&0\\
0&0&0&0&0&1&1&1&2&2&2&0&0&2&2&1
\end{array}\right]$$

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