All Self-Dual Z_4 Codes of Length 15 or Less Are Known

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Abstract

We classify the self-dual $\mathbb{Z}_4$ codes of all lengths from 10 through 15, extending the previous classification through length 9 [J.H. Conway, N.J.A. Sloane, Self-Dual Codes over the Integers Modulo 4, *J. Comb. Theory* **A 62** (1993), 31-45].

Keywords

Coding theory, Self-dual codes, Codes over rings, Classification.

I. Introduction

In [2] Conway and Sloane began the classification of self-dual $\mathbb{Z}_4$ codes. They classified all the codes up to length 9 without a "mass-formula". This formula was found by Gaborit [4] for both self-dual and Type II codes. Using this and an automorphism group program for $\mathbb{Z}_4$ codes, Pless, Leon and Fields [9] classified all of the Type II codes of length 16 (the next possible length after 8 where these codes exist). At the same time, Gaborit was pushing forward with the classification begun by Conway and Sloane. An unexpected consequence of the classification of the length 16 Type II codes was the determination of all of the length 15 self-dual codes by shortening the length 16 codes. Furthermore, all of the codes of lengths ≤ 14 appear as direct summands in the list of all length 15 codes.

In this paper we present both approaches to the classification of $\mathbb{Z}_4$ self-dual codes. In section 3, "The Bottom-Up Approach", the process of finding all of the self-dual $\mathbb{Z}_4$ codes by actually constructing them, is described. In section 4, "The Top-Down Approach", we discuss how the codes can be found by shortening the length 16 codes. Both methods are verified by orbit counting using the mass formula of [4]. The agreement of these two distinct approaches serves to confirm our results.

Appendix I lists all of the indecomposable codes of length 10 through 15, and gives the orders of their groups and partial symmetrized weight enumerators. The highest minimum Lee weight of a self-dual code of length 15 is 6. We give generator matrices for all such codes. Also, generator matrices for the indecomposable codes of lengths 10 and 11 are given. The highest minimum Lee weight attained in lengths 10 through 13 (and, with one exception, at length 14) is 4. Generator matrices for all of the codes are available on the world wide web at `http://www.math.uic.edu/~pless`.

II. Preliminaries

It follows from the discussion in [9] that a self-dual $\mathbb{Z}_4$ code C of length n has a generator matrix of the form:

$$G = \begin{bmatrix} A & (I_k + 2B) \\ & 2D \end{bmatrix} \tag{1}$$

where A, B and D are binary matrices. Further,

$$G_1 = \begin{bmatrix} A & I_k \end{bmatrix}$$

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generates a doubly-even binary $[n,k]$ code C_1 and

$$G_2 = \begin{bmatrix} A & I_k \\ & D \end{bmatrix}$$

generates $C_2 = C_1^{\perp}$.

In this situation C is said to have *type* $4^k 2^{n-2k}$ and indeed contains this many codewords.

The *symmetrized weight enumerator* of a $\mathbb{Z}_4$ code is a polynomial in two variables b and c such that the coefficient of $b^i c^j$ is the number of vectors in the code having exactly i 1's and 3's and j 2's.

Two $\mathbb{Z}_4$ codes are *equivalent* if one can be obtained from the other by a permutation of the coordinates possibly followed by multipying some coordinates by 3 (*sign changes*). The *group* $G(C)$ of a $\mathbb{Z}_4$ code C is the set of all coordinate permutations and sign changes that send the code onto itself. We denote by $\mathcal{M}_n$, the monomial group over $\mathbb{Z}_4$, which is the set of all sign-change permutations of n coordinates; it has order $n!2^n$. The index $|\mathcal{M}_n|/|G(C)|$ gives the number of codes equivalent to C.

We call a doubly-even binary code *admissible* if it contains the all-ones vector. From a given admissible code, one can construct $\mathbb{Z}_4$ codes for which the admissible code plays the role of C_1.

We label each binary doubly-even code with its length, dimension and a description of the subcodes generated by weight 4 vectors, using the standard notation d_{2m}, e_7 and e_8. Additionally, we use the label f in situations where C_1 does not contain any weight 4 vectors. To avoid confusing the binary and $\mathbb{Z}_4$ codes, we use a typographic convention: the labels of $\mathbb{Z}_4$ codes are rendered in bold face.

Note that it is possible for doubly-even codes of different dimensions to have similar labels since the labels make no mention of 'glue' vectors. Thus a length 15 doubly-even code with the label $[15,4]-2d_6$ has a generator matrix equivalent to the following:

$$\left[\begin{array}{cccccc|cccccc|ccc} 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&0&1&1&0&0&0&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&1&1&1&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&0&0&1&1&0&0&0 \end{array}\right]$$

Whereas the doubly-even code with the label $[15,5]-2d_6$ has the following generator matrix:

$$\left[\begin{array}{cccccc|cccccc|ccc} 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&0&1&1&0&0&0&0&0&0&0&0&0 \\ \hline 0&0&0&0&0&0&1&1&1&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&0&0&1&1&0&0&0 \\ \hline 1&0&1&0&1&0&1&0&1&0&1&0&1&1&0 \end{array}\right]$$

These labels completely distinguish the inequivalent binary doubly-even codes of lengths 10 through 16, however there are often inequivalent $\mathbb{Z}_4$ self-dual codes arising from the same C_1. These are further distinguished by appending a, b, ... to their labels.

If we have a length n $\mathbb{Z}_4$ code C, then the set of vectors in C which are either 0 or 2 on a particular coordinate is a subcode C'. Removing that coordinate from C' gives a code of length $n-1$ which is called the *shortened* code of C on that coordinate.

Lemma 1: If C is a Type II $\mathbb{Z}_4$ code of type $4^k 2^{n-2k}$ then shortening on any coordinate gives a self-dual code of type $4^{k-1} 2^{n-2k+1}$.

Proof: If we are shortening C on the jth coordinate, denote the shortened code by $S_j(C)$. Let us write $\tilde{h}$ for a vector consisting of all ones and threes. Since C is a Type II code, it contains a vector $\tilde{h}$. Thus for every coordinate position of C there are vectors that have 1's in that coordinate. Choose a basis for C such that exactly one basis vector has a 1 in the jth column, and all other basis vectors have 0's or 2's there. Without loss of generality, this vector is $\tilde{h}$.

In some sense, the subcode C' (the subcode which has only 0's and 2's in the jth coordinate) is of "co-dimension" 1. All of the other vectors (besides $\tilde{h}$) in the basis for C are in C'. Since we are considering a spanning set for C the vector $2\tilde{h}$ (i.e. the all-twos vector) is not in their span. To say that C has type $4^k 2^{n-2k}$, is to assert that every basis for C contains at least k vectors containing 1's and 3's. Clearly, we will have $k-1$ vectors having 1's and 3's in them in our basis for $S_j(C)$. Furthermore, by construction, this code is self-orthogonal and of length $n-1$. There were $n-2k$ vectors consisting solely of 2's and 0's in the basis for C, and we need to add the all-twos vector to get a complete basis for $S_j(C)$, so there are $n-2k+1$ vectors having only 2's and 0's in the basis for $S_j(C)$.

A self-dual code of length $n-1$ having at least $k-1$ vectors with 1's and/or 3's in any basis has type $4^{k-1}2^{(n-1)-2(k-1)}$. The shortened code we just constructed is self-orthogonal and has the right number of vectors, so it must be self-dual. ■

III. The Bottom-Up Approach

In this part we deal with the bottom up approach up to length 13, which consists in checking the mass formula of [4] by building the indecomposable self-dual $\mathbb{Z}_4$ codes of a given length n and then in completing them with decomposable codes obtained by direct sums of self-dual codes of length $n-1$ or less.

Theorem 5 of [4] shows us that in order to build self-dual codes over $\mathbb{Z}_4$, we first have to find all the inequivalent doubly even binary codes of a given length n. But self-dual $\mathbb{Z}_4$ codes whose projection onto one of the coordinate is not all of $\mathbb{Z}_4$ are decomposable [2], for they contain a subcode isomorphic to A_1 (the length one code composed of the two words '0' and '2'). Therefore, we only need to find the inequivalent doubly even binary codes of length 10 to 13 with no coordinate position identically zero.

Using mass formulas of [4], we build the classification of doubly even binary codes for length $10 \leq n \leq 13$, from which we extract the codes we are interested in. There are 21 such codes which are listed in Table I.

In order to classify doubly even binary codes, we first search for possible sub-codes generated by vectors of weight 4, which are obtained as direct sums of the standard codes d_{2m},e_7,e_8 and we eventually add 'glue' words of weight 8 to describe all the different codes. We know we have all the inequivalent codes, when the mass formula checks.

Code	Auto Order
[10,2]-d4	$2^7 3^2 5$
[10,3]-d6d4	$2^7 3^2$
[10,4]-d10	$2^8 3.5$
[11,3]-d6	$2^6 3^2 5$
[11,4]-e7d4	$2^6 3^2 7$
[12,1]-f	$2^{10} 3^3 5.7$
[12,2]-d4	$2^{10} 3^3 5.7$
[12,2]-f	$2^{10} 3^4$
[12,3]-d6	$2^8 3^3 5$
[12,3]-2d4	$2^{10} 3$

[12,3]-3d4	$2^{10} 3^4$
[12,4]-e7	$2^6 3^2 5.7$
[12,4]-d8	$2^9 3^2$
[12,4]-2d6	$2^9 3^2$
[12,4]-d8d4	$2^{10} 3^2$
[12,5]-d12	$2^{10} 3^2 5$
[12,5]-e8d4	$2^9 3^2 7$
[13,3]-d4	$2^6 3^4$
[13,4]-d6d4	$2^7 3^2$
[13,5]-d10	$2^8 3^2 5$
[13,5]-e7d6	$2^7 3^2 7$

Table I. List of the possible codes C_1, for lengths $10 \leq n \leq 13$.

In [4] we explained how to obtain all the self-dual Z_4 codes which have the same C_1.

Let C be a $\mathbb{Z}_4$ self-dual code with its associated binary codes C_1 and C_2 and let $(e_1,..,e_k)$ be a basis of C_1 which we complete to a basis $(e_1,..,e_n)$ of $\mathbb{F}_2^n$. Let us consider a k-set of vectors $(e_1^*,..,e_k^*)$, verifying

$e_i.e_j^* = \delta_{ij}$, the Kronecker symbol, $1 \le i,j \le k$. We know it is possible because of the existence of a dual basis of $(e_1,..,e_n)$.

We can associate to any self-dual $\mathbb{Z}_4$ code C a map f :

$$f : C_1 \longmapsto (\overline{e_1^*,..,e_k^*})$$

where $(\overline{e_1^*,..,e_k^*})$ is the subspace generated by the vectors $(e_1^*,..,e_k^*)$. Although f is not linear, it is determined by the image of a basis of C_1.

Let $M(m_{ij})_{1\le i,j\le k}$ be the matrix associated to f, $f(e_i) = \sum_{j=1}^{j=k_1} m_{ij}e_j^*$.

In the particular case of the generator matrix (1), where:

$$G_1 = \left[\begin{array}{cc} A & I_k \end{array} \right],$$

we can choose for the set $(e_1^*,..,e_k^*)$ the rows of the matrix:

$$Q = \left[\begin{array}{cc} 0 & I_k \end{array} \right]$$

which leads to M=B, with B the $k \times k$ matrix of the generator matrix of the self-dual code C (1).

In [4] we showed that the general way to build indecomposable Z_4 self-dual codes is to choose a C_1 in Table I, compute $C_2 = C_1^\perp$ and describe all the possible matrices M such that $e_i.e_j \equiv 2[m_{ij} + m_{ji}] \bmod 4$,$\forall\, 1 \le i,j \le k$. There are $2^{\frac{k(k+1)}{2}}$ such matrices, which represent the self-dual $\mathbb{Z}_4$ codes associated to a given C_1. Then we check if the code is really indecomposable by comparison with direct sums of self-dual codes of lower length.

In order to check the mass formula we have to compute the order of the automorphism groups of all the inequivalent self-dual codes we know for a given length. We are sure that our list is complete when it matches the mass formula.

The method described here differs from Leon's algorithm [6], which computes directly the automorphism groups. We have used a program computing the orbit $Orb_{(G)}C$ and the stabilizer $Aut_{(G)}C$ of a code C under the action of a given matrix group G. The advantage of such a program is that it can easily be carried out, but if we use it directly with $G = \mathcal{M}_n$ and a $\mathbb{Z}_4$ code C, in order to compute $|Aut_{(\mathcal{M}_n)}C| = \frac{|\mathcal{M}_n|}{|Orb_{(\mathcal{M}_n)}C|}$, we quickly get orbits with big size, which is too expensive in computation time.

To deal with this problem we define a matrix group $m_1 = \{\mathrm{diag}(a_1, a_2, \ldots, a_n)\ |a_i \in \{\mp 1\}\}$, and we notice that :

$$Aut_{(\mathcal{M}_n)}C = Aut_{(m_1 \rtimes Aut_{(S_n)}C_1)}C,$$

which leads to :

$$|Aut_{(\mathcal{M}_n)}C| = \frac{|Aut_{(S_n)}C_1| \times 2^n}{|Orb_{(m_1 \rtimes Aut_{(S_n)}C_1)}C|},$$

so that we only need to compute $Aut_{(S_n)}C_1$ and $Orb_{(m_1 \rtimes Aut_{(S_n)}C_1)}C$. We compute $Aut_{(S_n)}C_1$ once and when we change the matrix M, we compute and store the different elements of $Orb_{(m_1 \rtimes Aut_{(S_n)}C_1)}C$, in order to be sure that the next new matrix M is not associated to a code already found.

For example let us take C_1 = [13,5]-e7d6, with $k = 5$, given by a generator matrix G_1:

$$G_1 = \begin{bmatrix} 1&1&1&1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&1&0&0&0&0&0&0&0 \\ 1&0&1&0&1&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&1&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1&1&1&1 \end{bmatrix}$$

We compute the orbit and a set of generators of the automorphisms of C_1 under the action of S_{13}. We get 772200 equivalent codes and hence $\frac{13!}{772200}$ automorphisms. The code C_2 is given by $C_2 = C_1^{\perp}$ and a k_1-set $(e_1^*, .., e_k^*)$ associated with a matrix Q is deduced from the basis of C_1 :

$$\mathrm{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

We choose a matrix M among the $2^{\frac{5.6}{2}}$ possible matrices :

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix M, which characterizes the map F allows us to build a new self-dual code over $\mathbb{Z}_4$: **[13,5]-e7d6a**, with generator matrix obtained by $G_1 + 2MQ$ for the first k rows and by two times a supplement of C_1 in C_2 for the last $n - 2k$ rows (or the 2-part). When the matrix M changes, only the first k rows of the generator matrix change.

$$\begin{bmatrix} 1 & 3 & 1 & 3 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \end{bmatrix}$$

We compute the orbit of **[13,5]-e7d6a** under the action of $m_1 \rtimes Aut_{(S_n)}C_1$ and obtain 10752 equivalents from which we deduce that the order of the automorphism group of the $\mathbb{Z}_4$ code **[13,5]-e7d6a** under the action of $\mathcal{M}_n$ is

$$|Aut_{(\mathcal{M}_n)}[13,5] - e7d6a| = \frac{2^{13}.13!}{772200.10752} = 2^{11}.3$$

IV. The Top-Down Approach

A. The Length 15 Codes

Recall that the *Euclidean weight* of a $\mathbb{Z}_4$ vector v is an integer that counts a 2 component of v as 4, and a 1 or 3 component of v as 1. A *Type II code* is a self-dual code containing an $\tilde{h}$ vector in which all vectors have Euclidean weight divisible by 8. Type II codes exist only at lengths $n \equiv 0 \pmod 8$.

Note that any self-dual code of length 15 is a shortened code of a Type II length 16 code. By Lemma 1, shortening a Type II length 16 code does give a self-dual length 15 code. Conversely, given a generator matrix G' of a length 15 self-dual code, we can find a generator matrix G for a Type II length 16 code. Add a column to G', and fill in this column with 0's and 2's so that the Euclidean weights of all rows become $0 \pmod 8$. Add another row to G' consisting of a vector $\tilde{h}$ where the 1's and 3's in $\tilde{h}$ are adjusted so as to make it orthogonal to the extended G'.

To see how this works, consider the following generator matrix for $[\mathbf{16}, \mathbf{4}] - \mathbf{d_6}$.

$$G=\begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&3&3&1\\
1&1&1&0&0&0&0&0&0&0&0&0&2&1&0&0\\
1&1&0&1&0&0&0&0&0&0&0&0&0&2&1&0\\
1&0&1&1&1&1&1&1&0&0&0&0&0&2&2&1\\
2&2&0&0&2&0&0&0&2&0&0&0&0&0&0&0\\
2&0&2&2&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&0&2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&0&0&2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&2&0&0&0&0
\end{bmatrix}$$

Shortening on the 12th coordinate we obtain

$$S_{12}(G)=\begin{bmatrix}
1&1&1&0&0&0&0&0&0&0&0&2&1&0&0\\
1&1&0&1&0&0&0&0&0&0&0&0&2&1&0\\
1&0&1&1&1&1&1&1&0&0&0&0&2&2&1\\
2&2&0&0&2&0&0&0&2&0&0&0&0&0&0\\
2&0&2&2&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&0&2&0&0&0&0&0&0&0&0\\
0&0&0&0&2&0&0&2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&2&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&0\\
2&2&2&2&2&2&2&2&2&2&2&2&2&2&2
\end{bmatrix}$$

a generator matrix for $[\mathbf{15},\mathbf{3}]-\mathbf{d_6}$. It is clear how to add a column and a row equivalent to the all-ones vector to $S_{12}(G)$ and get a generator matrix for a Type II code. No matter how we do this, we get a generator matrix for a code equivalent to the one we started with.

Theorem 1: There are 290 inequivalent self-dual $\mathbb{Z}_4$ codes of length 15, 111 of them are indecomposable. The highest minimum Lee weight is 6, and there are 14 codes attaining this minimum Lee weight. The number of codes of each type 4^k2^{15-2k} are as follows:

k	0	1	2	3	4	5	6	7
number	1	3	9	26	58	86	77	30

These codes were found by shortening each of the 133 length 16 Type II codes on coordinates in different orbits of the groups of these codes. That these 290 codes represent all of the inequivalent length 15 self-dual codes has been verified using the mass formula from [4].

In Appendix I we give the labels of the indecomposable self-dual codes of length 15, their group orders (the group orders are factored as (order of the sign change group) $\times$ (order of the permutation group)), and partial information about their symmetrized weight enumerators.

The highest minimum Lee weight is 6, and there are 14 codes with this minimum weight. Five of them are shortened from the five Type II codes having minimum Lee weight 8, the rest are shortened from some of the Type II codes of minimum Lee weight 6. We give the generator matrices of the codes of minimum Lee weight 6 in Appendix II.

The Gray map of $[\mathbf{15},\mathbf{4}]-\mathbf{f}$ gives the linear singly-even [30,15,6] code, a child of the doubly-even code $R(1,5)$.

There is a relation between the number of Type II codes of length 16 and the number of self-dual codes of length 15. We end this section by clarifying this relation.

Theorem 2: Let n be divisible by 8. Let $\tau(n,k)$ denote the number of admissible binary codes of length n and dimension k, and $\sigma(n-1,k-1)$ the number of doubly-even $[n-1,k-1]$ codes. Then

$$\tau(n,k)=\sigma(n-1,k-1).$$

Further, the number of Type II codes of length n and type 4^k2^{n-2k} is twice the number of $\mathbb{Z}_4$ self-dual codes of length $n-1$ and type $4^{k-1}2^{n-2k+2}$.

Proof: The statement for the binary codes can be demonstrated in two ways, one way is by a combinatorial proof and the other uses formulas.

Given an $[n-1,k-1]$ doubly-even code C', adjoining a new coordinate, say the last one, on which C' is zero, and adding the all-ones vector produces an admissible $[n,k]$ code. Conversely, shortening an admissible $[n,k]$ code on, say, its last coordinate produces a doubly-even $[n-1,k-1]$ code.

For the other proof, note that $\tau(n,k)$ satisfies the following recursion [7]

$$\tau(n,1)=1$$

and

$$\tau(n,k)=\tau(n,k-1)\frac{2^{n-2k+1}+2^{n/2-k}-1}{2^{k-1}-1}.$$

Also, $\sigma(n-1,k-1)$ satisfies the same recursion and $\sigma(n-1,0)=1$.

Now the number of Type II codes of length n and type 4^k2^{n-2k} [4] is $\tau(n,k)2^{k(k-1)/2+1}$. From the same reference, the number of self-dual codes of length $n-1\equiv 7 \pmod 8$ and type $4^{k-1}2^{n-2k+1}$ is

$$\sigma(n-1,k-1)2^{k(k-1)/2}.$$

This demonstrates the second statement in the theorem.

This can also be demonstrated combinatorially by noting that there are two ways to extend a self-dual $\mathbb{Z}_4$ code of length $n-1$ and type $4^{k-1}2^{n-2k+1}$ to a Type II code of length n and type 4^k2^{n-2k} which correspond to the two possibilities (1 or 3) for the new column in the new vector $\tilde{h}$. ∎

B. Lengths 10,11,12,13 and 14

Lemma 2: If C contains a self-dual $\mathbb{Z}_4$ code of length n whose symmetrized weight enumerator contains the term rc, the code obtained by puncturing the r columns of a generator matrix for C that consist entirely of 0's and 2's gives a generator matrix of a self-dual $\mathbb{Z}_4$ code C' of length $n-r$.

Further C' is either indecomposable or is the direct sum of indecomposable codes each of which is length 4 or more.

Proof: From the data in [2], 4 is the smallest length of an indecomposable code whose length is greater than 1. ∎

From the lemma, it is clear that all the information about indecomposable codes of lengths 10,11,12,13 and 14 is contained in the data on all of the length 15 self-dual codes and the data in [2] about indecomposable codes of smaller lengths (4,6,7 and 8).

Theorem 3: The number of indecomposable codes of length $10 \le n \le 14$ and type $4^k 2^{n-2k}$ are:

$n \backslash k$	1	2	3	4	5	6
10	0	1	1	2		
11	0	0	1	1		
12	1	2	6	11	11	
13	0	0	1	2	4	
14	0	2	6	20	32	32

We punctured all of the length 15 codes which had $5c$ in their symmetrized weight enumerators obtaining all of the length 10 indecomposable codes and one direct sum $[\mathbf{4}, \mathbf{1}] - \mathbf{d4} \oplus [\mathbf{6}, \mathbf{2}] - \mathbf{d6}$. We checked that we had indeed found all the codes by finding their group orders, finding all direct sums involving indecomposable summands of lengths 4,6,7 and 8 and their group orders and then using the mass-formula.

We did similar computations for length 11 using length 15 self-dual codes whose symmetrized weight enumerators contained the term $4c$. We found all direct sums of codes of shorter lengths from the data in [2] and the indecomposable codes of length 10. All group orders were computed and the mass formula checked. Similar computations were done in turn for lengths 12, 13 and 14, using length 15 self-dual codes whose symmetrized weight enumerators contained the terms $3c$, $2c$ and $1c$ respectively. Again, all direct sums of the appropriate lengths were constructed using the data from [2] and our newfound knowledge of the indecomposable codes of lengths between 10 and the length in question.

We give the labels, group orders, and partial information about symmetrized weight enumerators of the self-dual codes having no trivial component of length n, for $10 \le n \le 14$, in Appendix I.

We give the generator matrices of the indecomposable codes of lengths 10 and 11 in Appendix II. There are too many indecomposable codes of lengths $12 \le n \le 15$ for inclusion here, so we give generator matrices only for those indecomposable codes (one of length 14 and fifteen of length 15) having minimum Lee weight 6.

V. Appendix I: Data For All Self-Dual Codes (without Trivial Components) of Lengths 10,11,12,13,14 & 15

Most of the codes below have minimum Lee weight 4,the few exceptions to this rule (which have minimum Lee weight 6) are marked with an asterisk.

A. *LENGTH 10*

name	group order (sign change)perm.	orbit lengths	swe coefficients						
			c^2	c^3	c^4	b^4	b^4c^1	b^4c^2	b^8
[10,2]-d4	$(2^7)2^7 3^2 5$	4,6	17	24	46	4	24	60	128
[10,3]-d4⊕d6	$(2^6)2^7 3^2$	4,6	9	8	22	20	24	108	96
[10,3]-d4d6	$(2^6)2^7 3$	2,4,4	9	8	22	4	56	124	96
[10,4]-d10a	$(2^5)2^6 5$	10	5	0	10	20	40	140	80
[10,4]-d10b	$(2^5)2^6 3$	4,6	5	0	10	4	72	156	80

B. *LENGTH 11*

name	group order (sign change)perm.	orbit lengths	swe coefficients						
			c^2	c^3	c^4	b^4	b^4c^1	b^4c^2	b^8
[11,3]-d6	$(2^5)2^6 3^2 5$	5,6	10	19	28	6	42	126	128
[11,4]-d4⊕e7	$(2^4)2^6 3^2 7$	4,7	6	7	8	22	42	126	112
[11,4]-d4e7	$(2^4)2^6 3^2$	1,4,6	6	7	8	6	58	174	112

C. *LENGTH 12*

name	group order (sign change)perm.	orbit lengths	swe coefficients						
			c^2	c^3	c^4	b^4	b^4c^1	b^4c^2	b^8
[12,1]-f	$(2^{11})2^{10} 3^5 5^2 7 \cdot 11$	12	66	0	495	0	0	0	0
[12,2]-d4⊕f(8)	$(2^{10})2^{10} 3^3 5 \cdot 7$	4,8	34	0	239	8	0	224	128
[12,2]-d4	$(2^{10})2^{10} 3^3 5 \cdot 7$	4,8	34	0	239	0	64	0	0
[12,2]-f	$(2^9)2^{10} 3^4$	12	18	64	111	0	0	0	192
[12,3]-2d4a	$(2^7)2^{10} 3$	4,8	10	32	47	8	32	96	160
[12,3]-2d4b	$(2^7)2^{10} 3$	4,8	10	32	47	0	32	128	160
[12,3]-d4⊕d4⊕d4	$(2^9)2^{10} 3^4$	12	18	0	111	24	0	288	192
[12,3]-3d4a	$(2^9)2^{10} 3^3$	4,8	18	0	111	0	64	128	64
[12,3]-3d4b	$(2^9)2^{10} 3^4$	12	18	0	111	0	0	384	192
[12,3]-d4⊕2d4	$(2^9)2^{10} 3^3$	4,8	18	0	111	8	64	96	64
[12,3]-d6a	$(2^8)2^8 3^3 5$	6,6	18	0	111	0	96	0	0
[12,3]-d6b	$(2^8)2^8 3^2 5$	2,4,6	18	0	111	8	32	224	128
[12,4]-d4⊕d8II	$(2^7)2^{10} 3^2$	4,8	10	0	47	8	96	32	32
[12,4]-d4⊕d8I	$(2^7)2^{10} 3$	4,8	10	0	47	24	32	224	160
[12,4]-d4d8a	$(2^7)2^{10} 3$	4,8	10	0	47	0	96	64	32
[12,4]-d4d8b	$(2^7)2^8 3$	2,2,4,4	10	0	47	8	64	160	96
[12,4]-d4d8c	$(2^7)2^9 3$	4,8	10	0	47	8	32	288	160
[12,4]-d4d8e	$(2^7)2^8 3^2$	2,4,6	10	0	47	0	64	192	96
[12,4]-d4d8f	$(2^7)2^{10} 3$	4,8	10	0	47	0	32	320	160
[12,4]-d6⊕d6	$(2^6)2^9 3^2$	12	6	16	15	24	48	96	144
[12,4]-d6d6a	$(2^6)2^9$	4,8	6	16	15	8	48	160	144
[12,4]-d6d6b	$(2^6)2^8 3$	12	6	16	15	0	48	192	144
[12,4]-d8a	$(2^4)2^9 3$	4,8	6	16	15	8	48	160	144

name	group order	orbit	swe coefficients						
	(sign change)perm.	lengths	c^2	c^3	c^4	b^4	b^4c^1	b^4c^2	b^8
[12,4]-d8b	$(2^4)2^9 3^2$	4,8	6	16	15	0	48	192	144
[12,4]-e7a	$(2^5)2^6 3^2 5\cdot 7$	5,7	10	0	47	0	112	0	0
[12,4]-e7b	$(2^5)2^6 3^2 5$	1,5,6	10	0	47	8	48	224	128
[12,5]-d12a	$(2^6)2^8 3$	12	6	0	15	24	48	192	144
[12,5]-d12b	$(2^6)2^7 3$	2,4,6	6	0	15	8	80	128	80
[12,5]-d12c	$(2^6)2^8$	4,8	6	0	15	8	48	256	144
[12,5]-d12d	$(2^6)2^7 5$	2,10	6	0	15	0	80	160	80
[12,5]-d12e	$(2^6)2^8 3^2$	12	6	0	15	0	48	288	144
[12,5]-d12f	$(2^6)2^8 3$	12	6	0	15	0	48	288	144
[12,5]-d4⊕e8II	$(2^4)2^9 3^2 7$	4,8	6	0	15	8	112	0	16
[12,5]-d4⊕e8I	$(2^4)2^9 3^2$	4,8	6	0	15	24	48	192	144
[12,5]-d4e8a	$(2^4)2^9 3^2$	4,8	6	0	15	0	112	32	16
[12,5]-d4e8b	$(2^4)2^6 3^2$	1,1,4,6	6	0	15	8	64	192	112
[12,5]-d4e8c	$(2^4)2^8 3$	4,8	6	0	15	8	48	256	144
[12,5]-d4e8e	$(2^4)2^6 3^2 7$	1,4,7	6	0	15	0	64	224	112
[12,5]-d4e8f	$(2^4)2^9 3^2$	4,8	6	0	15	0	48	288	144

D. LENGTH 13

name	group order	orbit	swe coefficients						
	(sign change)perm.	lengths	c^2	c^3	c^4	b^4	b^4c^1	b^4c^2	b^8
[13,3]-d4	$(2^7)2^6 3^4$	4,9	9	45	82	2	18	72	192
[13,4]-d4⊕d6a	$(2^5)2^7 3^2$	3,4,6	5	25	34	10	42	120	176
[13,4]-d4⊕d6b	$(2^5)2^7 3$	2,3,4,4	5	25	34	2	34	152	176
[13,5]-d10a	$(2^3)2^6 3\cdot 5$	3,10	3	15	10	10	50	160	168
[13,5]-d10b	$(2^3)2^6 3^2$	3,4,6	3	15	10	2	42	192	168
[13,5]-d6⊕e7	$(2^4)2^7 3^2 7$	6,7	3	15	10	26	66	96	168
[13,5]-d6e7b	$(2^4)2^7 3$	1,2,4,6	3	15	10	10	50	160	168
[13,5]-d6e7c	$(2^4)2^6 3$	3,4,6	3	15	10	2	42	192	168

E. LENGTH 14

name	group order	orbit	swe coefficients						
	(sign change)perm.	lengths	c^2	c^3	c^4	b^4	b^4c^1	b^4c^2	b^8
[14,2]-d4	$(2^{11})2^{11} 3^4 5^2 7$	4,10	47	40	301	4	40	180	0
[14,2]-f	$(2^{11})2^{10} 3^4 5^2$	2,12	31	72	285	0	0	0	128
[14,3]-d4⊕d4(10)	$(2^{10})2^{10} 3^3 5$	4,4,6	23	24	149	12	24	220	160
[14,3]-d4a	$(2^9)2^{11} 3^2$	2,4,8	15	40	141	0	32	64	128
[14,3]-d4b	$(2^9)2^{11} 3^2$	2,4,8	15	40	141	4	8	116	192
[14,3]-d4+d4(10)a	$(2^{10})2^{10} 3^3 5$	4,4,6	23	24	149	4	72	116	32
[14,3]-d4+d4(10)b	$(2^{10})2^9 3^3 5$	2,2,4,6	23	24	149	0	32	192	96
[14,3]-d6⊕f8	$(2^{10})2^{11} 3^3 5\cdot 7$	6,8	31	8	157	12	24	348	128
[14,3]-d6a	$(2^{10})2^{11} 3^2 5\cdot 7$	2,4,8	31	8	157	4	72	244	0
[14,3]-f	$(2^8)2^{10} 3\cdot 7$	14	7	56	133	0	0	0	224
[14,4]-d4⊕d4⊕d6	$(2^9)2^{11} 3^3$	6,8	15	8	77	28	24	300	256
[14,4]-2d4a	$(2^5)2^9 3$	6,8	3	32	65	4	24	84	208

name	group order (sign change)perm.	orbit lengths	swe coefficients c^2	c^3	c^4	b^4	b^4c^1	b^4c^2	b^8
[14,4]-2d4b	$(2^5)2^9 3$	6,8	3	32	65	0	16	96	208
[14,4]-2d4d6a	$(2^9)2^{10}3^2$	2,4,4,4	15	8	77	4	72	180	64
[14,4]-2d4d6b	$(2^9)2^{10}3^2$	2,4,8	15	8	77	0	64	192	64
[14,4]-2d4d6c	$(2^9)2^{10}3^2$	2,4,8	15	8	77	0	32	256	128
[14,4]-2d4d6d	$(2^9)2^{11}3^2$	2,4,8	15	8	77	4	40	244	128
[14,4]-2d4d6e	$(2^9)2^{11}3^2$	2,4,8	15	8	77	4	8	308	192
[14,4]-3d4a	$(2^7)2^{11}$	2,4,8	7	24	69	0	32	128	160
[14,4]-3d4b	$(2^7)2^{11}3$	2,12	7	24	69	12	24	156	224
[14,4]-3d4c	$(2^7)2^{11}$	2,4,8	7	24	69	4	40	116	160
[14,4]-3d4d	$(2^7)2^{11}3$	2,12	7	24	69	0	0	192	224
[14,4]-d4⊕d4d6	$(2^9)2^{10}3^2$	2,4,4,4	15	8	77	12	56	220	128
[14,4]-d4d6a	$(2^8)2^{10}3^2$	4,4,6	11	16	73	4	88	84	48
[14,4]-d4d6b	$(2^8)2^9 3$	2,2,2,4,4	11	16	73	0	48	160	112
[14,4]-d4d6c	$(2^8)2^{10}3$	2,4,4,4	11	16	73	12	40	188	176
[14,4]-d4d6d	$(2^8)2^9 3$	2,4,4,4	11	16	73	4	24	212	176
[14,4]-d6⊕2d4	$(2^9)2^{11}3^3$	6,8	15	8	77	12	88	156	64
[14,4]-d6a	$(2^6)2^7 3^3$	2,6,6	7	24	69	0	48	96	128
[14,4]-d6b	$(2^6)2^7 3^2$	2,2,4,6	7	24	69	4	24	148	192
[14,4]-d8a	$(2^9)2^9 3^2 5$	4,4,6	19	0	81	4	88	212	16
[14,4]-d8b	$(2^9)2^8 3^3 5$	2,6,6	19	0	81	0	48	288	80
[14,4]-d8c	$(2^9)2^9 3^2 5$	4,4,6	19	0	81	12	40	316	144
[14,5]-2d6a	$(2^3)2^8 3^2$	2,12	1	20	31	12	48	108	200
[14,5]-2d6b	$(2^3)2^8$	2,4,8	1	20	31	4	32	132	200
[14,5]-2d6c	$(2^3)2^7 3$	2,12	1	20	31	0	24	144	200
[14,5]-d4⊕d10a	$(2^8)2^9 3\cdot 5$	4,10	11	0	41	28	40	300	240
[14,5]-d4⊕d10b	$(2^8)2^9 3^2$	4,4,6	11	0	41	12	72	220	112
[14,5]-d4d10a	$(2^8)2^9 3$	2,4,4,4	11	0	41	4	88	180	48
[14,5]-d4d10b	$(2^8)2^8 3^2$	2,2,4,6	11	0	41	0	80	192	48
[14,5]-d4d10c	$(2^8)2^9 3$	2,4,4,4	11	0	41	12	72	220	112
[14,5]-d4d10d	$(2^8)2^8 3$	2,2,2,2,2,4	11	0	41	0	48	256	112
[14,5]-d4d10e	$(2^8)2^9 3$	2,4,4,4	11	0	41	12	40	284	176
[14,5]-d4d10f	$(2^8)2^9 3$	2,4,4,4	11	0	41	4	56	244	112
[14,5]-d4d10g	$(2^8)2^9 3^2$	4,4,6	11	0	41	4	24	308	176
[14,5]-d4d10h	$(2^8)2^9 3$	2,4,4,4	11	0	41	4	24	308	176
[14,5]-d4d8a	$(2^4)2^{10}$	2,4,8	3	16	33	0	48	128	144
[14,5]-d4d8b	$(2^4)2^{10}$	2,4,8	3	16	33	12	40	156	208
[14,5]-d4d8c	$(2^4)2^8$	2,2,2,4,4	3	16	33	4	40	148	176
[14,5]-d4d8d	$(2^4)2^9$	2,4,8	3	16	33	4	24	180	208
[14,5]-d4d8e	$(2^4)2^{10}3$	2,4,8	3	16	33	4	56	116	144
[14,5]-d4d8f	$(2^4)2^8 3$	2,2,4,6	3	16	33	0	32	160	176
[14,5]-d4d8g	$(2^4)2^{10}$	2,4,8	3	16	33	0	16	192	208
[14,5]-d4e7a	$(2^5)2^7 3^2 7$	3,4,7	5	12	35	4	96	68	56
[14,5]-d4e7b	$(2^5)2^6 3^2$	1,2,2,3,6	5	12	35	0	56	144	120

name	group order	orbit	swe coefficients						
	(sign change)perm.	lengths	c^2	c^3	c^4	b^4	b^4c^1	b^4c^2	b^8
[14,5]-d4e7c	$(2^5)2^7 3^2$	1,3,4,6	5	12	35	12	48	172	184
[14,5]-d4e7d	$(2^5)2^6 3$	1,2,3,4,4	5	12	35	4	32	196	184
[14,5]-d6⊕d8I	$(2^7)2^{11} 3$	6,8	7	8	37	28	56	204	224
[14,5]-d6⊕d8II	$(2^7)2^{11} 3^2$	6,8	7	8	37	12	120	60	32
[14,5]-d6d8a	$(2^7)2^9 3$	2,2,4,6	7	8	37	4	72	148	96
[14,5]-d6d8b	$(2^7)2^{11}$	2,4,8	7	8	37	4	72	148	96
[14,5]-d6d8c	$(2^7)2^9$	2,4,4,4	7	8	37	0	64	160	96
[14,5]-d6d8d	$(2^7)2^9$	2,2,2,4,4	7	8	37	12	56	188	160
[14,5]-d6d8e	$(2^7)2^9$	2,4,4,4	7	8	37	0	32	224	160
[14,5]-d6d8f	$(2^7)2^{10}$	2,4,8	7	8	37	0	32	224	160
[14,5]-d6d8g	$(2^7)2^{10}$	2,4,8	7	8	37	12	24	252	224
[14,5]-d6d8h	$(2^7)2^8$	2,4,4,4	7	8	37	4	40	212	160
[14,5]-d6d8i	$(2^7)2^{11}$	2,4,8	7	8	37	4	40	212	160
[14,5]-d6d8j	$(2^7)2^{10} 3$	6,8	7	8	37	0	0	288	224
[14,6]-e7⊕e7	$(2^2)2^7 3^2 7^2$	14	0	14	14	28	84	84	196
[14,6]-2e7a	$(2^2)2^7 3^2$	2,12	0	14	14	12	52	132	196
[14,6]-2e7b	$(2^2)2^6 3$	6,8	0	14	14	4	36	156	196
* **[14,6]-2e7c**	$(2^2)2^4 3\cdot 7$	14	0	14	14	0	28	168	196
[14,6]-d12a	$(2^2)2^8 3$	2,12	1	12	15	12	48	156	200
[14,6]-d12b	$(2^2)2^7 3$	2,2,4,6	1	12	15	4	48	148	168
[14,6]-d12c	$(2^2)2^8$	2,4,8	1	12	15	4	32	180	200
[14,6]-d12d	$(2^2)2^7 5$	2,2,10	1	12	15	0	40	160	168
[14,6]-d12e	$(2^2)2^8 3^2$	2,12	1	12	15	0	24	192	200
[14,6]-d12f	$(2^2)2^8 3$	2,12	1	12	15	0	24	192	200
[14,6]-d14a	$(2^7)2^8$	2,4,4,4	7	0	21	4	72	196	96
[14,6]-d14b	$(2^7)2^8 7$	14	7	0	21	28	56	252	224
[14,6]-d14c	$(2^7)2^8 3^2$	2,12	7	0	21	0	96	144	32
[14,6]-d14d	$(2^7)2^8 3$	4,4,6	7	0	21	12	88	172	96
[14,6]-d14e	$(2^7)2^8$	2,4,4,4	7	0	21	0	64	208	96
[14,6]-d14f	$(2^7)2^8$	2,4,4,4	7	0	21	12	56	236	160
[14,6]-d14g	$(2^7)2^8$	2,4,4,4	7	0	21	0	32	272	160
[14,6]-d14h	$(2^7)2^8 3$	2,12	7	0	21	12	24	300	224
[14,6]-d14i	$(2^7)2^8 3$	4,4,6	7	0	21	4	40	260	160
[14,6]-d14j	$(2^7)2^8 5$	4,10	7	0	21	4	40	260	160
[14,6]-d14k	$(2^7)2^8$	2,4,4,4	7	0	21	4	40	260	160
[14,6]-d14l	$(2^7)2^8 3\cdot 7$	14	7	0	21	0	0	336	224
[14,6]-d6⊕e8I	$(2^4)2^{10} 3^2$	6,8	3	8	17	28	72	156	208
[14,6]-d6⊕e8II	$(2^4)2^{10} 3^2 7$	6,8	3	8	17	12	136	12	16
[14,6]-d6e8a	$(2^4)2^{10} 3$	2,4,8	3	8	17	4	88	100	80
[14,6]-d6e8b	$(2^4)2^7 3$	2,2,4,6	3	8	17	0	64	144	112
[14,6]-d6e8c	$(2^4)2^7 3$	1,1,2,4,6	3	8	17	12	56	172	176
[14,6]-d6e8d	$(2^4)2^9$	2,4,8	3	8	17	0	48	176	144
[14,6]-d6e8e	$(2^4)2^9$	2,4,8	3	8	17	12	40	204	208

name	group order (sign change)perm.	orbit lengths	swe coefficients						
			c^2	c^3	c^4	b^4	b^4c^1	b^4c^2	b^8
[14,6]-d6e8f	$(2^4)2^6$	2,2,2,4,4	3	8	17	4	40	196	176
[14,6]-d6e8g	$(2^4)2^7 3$	6,8	3	8	17	4	24	228	208
[14,6]-d6e8h	$(2^4)2^7 3 \cdot 7$	1,2,4,7	3	8	17	4	72	132	112
[14,6]-d6e8i	$(2^4)2^{10} 3$	2,4,8	3	8	17	4	56	164	144
[14,6]-d6e8j	$(2^4)2^7 3$	2,2,4,6	3	8	17	0	32	208	176
[14,6]-d6e8k	$(2^4)2^9 3$	6,8	3	8	17	0	16	240	208

F. LENGTH 15

name	group order (sign change)perm.	orbit lengths	swe coefficients						
			c^2	c^3	c^4	b^4	b^4c^1	b^4c^2	b^8
[15,3]-d4	$(2^9)2^9 3^2 5^2$	1,4,10	20	47	181	2	22	110	128
[15,3]-d6	$(2^9)2^{10} 3^5 5 \cdot 7$	6,9	36	31	165	6	66	330	0
[15,3]-f	$(2^9)2^7 3^5$	3,12	12	55	189	0	0	0	192
[15,4]-2d4a	$(2^7)2^8 3^2$	1,4,4,6	8	27	97	2	38	94	144
[15,4]-2d4b	$(2^7)2^7 3^2$	1,2,2,4,6	8	27	97	0	16	112	176
[15,4]-2d4c	$(2^7)2^8 3^2$	1,4,4,6	8	27	97	6	18	122	208
[15,4]-d4⊕d6(11)	$(2^8)2^9 3^3 5$	4,5,6	16	19	89	14	42	242	176
[15,4]-d4a	$(2^6)2^9 3$	3,4,8	4	31	101	0	16	48	192
[15,4]-d4b	$(2^6)2^9 3$	3,4,8	4	31	101	2	6	62	224
[15,4]-d4d6a	$(2^8)2^9 3^3 5$	4,5,6	16	19	89	6	82	186	48
[15,4]-d4d6b	$(2^8)2^8 3^2 5$	1,1,4,4,5	16	19	89	2	38	222	112
[15,4]-d6a	$(2^7)2^{10} 3^2$	1,2,4,8	12	23	93	2	38	158	128
[15,4]-d6b	$(2^7)2^{10} 3^3$	1,6,8	12	23	93	6	18	186	192
[15,4]-e7	$(2^8)2^{10} 3^3 5 \cdot 7$	1,6,8	28	7	77	6	82	378	0
[15,4]-e7⊕2d4	$(2^7)2^{10} 3^3 7$	7,8	12	7	45	14	106	210	64
[15,4]-e7⊕f8	$(2^8)2^{10} 3^3 5 \cdot 7^2$	7,8	28	7	77	14	42	434	128
* **[15,4]-f**	$(2^5)2^6 3^2 5 \cdot 7$	15	0	35	105	0	0	0	240
[15,5]-2d4d6a	$(2^5)2^9$	1,2,4,4,4	4	15	53	2	38	126	160
[15,5]-2d4d6b	$(2^5)2^9$	1,2,4,8	4	15	53	0	32	128	160
[15,5]-2d4d6c	$(2^5)2^9$	1,2,4,4,4	4	15	53	6	34	138	192
[15,5]-2d4d6d	$(2^5)2^9$	1,2,4,8	4	15	53	0	16	144	192
[15,5]-2d4d6e	$(2^5)2^{10} 3$	1,6,8	4	15	53	6	50	122	160
[15,5]-2d4d6f	$(2^5)2^{10}$	1,2,4,8	4	15	53	2	22	142	192
[15,5]-2d4d6g	$(2^5)2^{10} 3$	1,6,8	4	15	53	14	26	162	256
[15,5]-2d4d6h	$(2^5)2^{10}$	1,2,4,8	4	15	53	2	6	158	224
[15,5]-2d4e7a	$(2^7)2^9 3^3$	1,4,4,6	12	7	45	6	82	218	64
[15,5]-2d4e7b	$(2^7)2^9 3^2$	1,2,4,8	12	7	45	2	54	238	96
[15,5]-2d4e7c	$(2^7)2^9 3^2$	1,2,4,8	12	7	45	2	38	254	128
[15,5]-2d4e7d	$(2^7)2^{10} 3^3$	1,6,8	12	7	45	6	34	266	160
[15,5]-2d4e7e	$(2^7)2^{10} 3^3$	1,6,8	12	7	45	6	18	282	192
[15,5]-2d6a	$(2^6)2^8 3^3$	3,6,6	6	13	51	6	90	114	72
[15,5]-2d6b	$(2^6)2^8 3^2$	2,3,4,6	6	13	51	14	50	170	200
[15,5]-2d6c	$(2^6)2^7 3$	1,1,2,3,4,4	6	13	51	2	46	150	136
[15,5]-2d6d	$(2^6)2^5 3^2$	3,3,3,6	6	13	51	0	24	168	168

	name	group order	orbit	swe coefficients						
		(sign change)perm.	lengths	c^2	c^3	c^4	b^4	b^4c^1	b^4c^2	b^8
	[15,5]-2d6e	$(2^6)2^7 3$	2,2,3,4,4	6	13	51	6	26	178	200
*	**[15,5]-3d4a**	$(2^3)2^8 3$	3,4,8	0	19	57	0	16	80	208
	[15,5]-3d4b	$(2^3)2^8 3^2$	3,12	0	19	57	6	18	90	240
	[15,5]-3d4c	$(2^3)2^8 3$	3,4,8	0	19	57	2	22	78	208
*	**[15,5]-3d4d**	$(2^3)2^8 3^2$	3,12	0	19	57	0	0	96	240
	[15,5]-d4⊕d4⊕e7	$(2^7)2^{10} 3^3 7$	7,8	12	7	45	30	42	306	288
	[15,5]-d4⊕d4e7(11)	$(2^7)2^9 3^3$	1,4,4,6	12	7	45	14	58	258	160
	[15,5]-d4d6a	$(2^4)2^8 3$	1,4,4,6	2	17	55	2	46	86	152
	[15,5]-d4d6b	$(2^4)2^7$	1,2,2,2,4,4	2	17	55	0	24	104	184
	[15,5]-d4d6c	$(2^4)2^8$	1,2,4,4,4	2	17	55	6	26	114	216
	[15,5]-d4d6d	$(2^4)2^7$	1,2,4,4,4	2	17	55	2	14	118	216
	[15,5]-d8a	$(2^5)2^7 3^2$	1,4,4,6	6	13	51	2	46	150	136
	[15,5]-d8b	$(2^5)2^6 3^3$	1,2,6,6	6	13	51	0	24	168	168
	[15,5]-d8c	$(2^5)2^7 3^2$	1,4,4,6	6	13	51	6	26	178	200
	[15,6]-d4d10a	$(2^3)2^8$	1,2,4,4,4	2	9	31	2	46	134	152
	[15,6]-d4d10b	$(2^3)2^7 3$	1,2,2,4,6	2	9	31	0	40	136	152
	[15,6]-d4d10c	$(2^3)2^8 3$	1,4,4,6	2	9	31	6	42	146	184
	[15,6]-d4d10d	$(2^3)2^8$	1,2,4,4,4	2	9	31	6	42	146	184
	[15,6]-d4d10e	$(2^3)2^7$	1,2,2,2,2,2,4	2	9	31	0	24	152	184
	[15,6]-d4d10f	$(2^3)2^8$	1,2,4,4,4	2	9	31	6	26	162	216
	[15,6]-d4d10g	$(2^3)2^8$	1,2,4,4,4	2	9	31	2	30	150	184
	[15,6]-d4d10h	$(2^3)2^8 5$	1,4,10	2	9	31	14	34	170	248
	[15,6]-d4d10i	$(2^3)2^8 3$	1,4,4,6	2	9	31	2	14	166	216
	[15,6]-d4d10j	$(2^3)2^8$	1,2,4,4,4	2	9	31	2	14	166	216
	[15,6]-d6d8a	$(2^2)2^7 3$	1,2,2,4,6	0	11	33	2	38	110	176
	[15,6]-d6d8b	$(2^2)2^9 3$	1,6,8	0	11	33	14	42	130	240
	[15,6]-d6d8c	$(2^2)2^9$	1,2,4,8	0	11	33	2	38	110	176
*	**[15,6]-d6d8d**	$(2^2)2^7$	1,2,4,4,4	0	11	33	0	32	112	176
	[15,6]-d6d8e	$(2^2)2^7$	1,2,2,2,4,4	0	11	33	6	34	122	208
*	**[15,6]-d6d8f**	$(2^2)2^7$	1,2,4,4,4	0	11	33	0	16	128	208
*	**[15,6]-d6d8g**	$(2^2)2^8$	1,2,4,8	0	11	33	0	16	128	208
	[15,6]-d6d8h	$(2^2)2^8$	1,2,4,8	0	11	33	6	18	138	240
	[15,6]-d6d8i	$(2^2)2^6$	1,2,4,4,4	0	11	33	2	22	126	208
	[15,6]-d6d8j	$(2^2)2^9 3^2$	1,6,8	0	11	33	6	66	90	144
	[15,6]-d6d8k	$(2^2)2^9$	1,2,4,8	0	11	33	2	22	126	208
*	**[15,6]-d6d8l**	$(2^2)2^8 3$	1,6,8	0	11	33	0	0	144	240
	[15,6]-d6e7a	$(2^3)2^7 3^2 7$	2,6,7	1	10	32	6	94	78	84
	[15,6]-d6e7b	$(2^3)2^6 3$	1,1,1,2,4,6	1	10	32	2	50	114	148
	[15,6]-d6e7c	$(2^3)2^7 3^2$	1,2,6,6	1	10	32	14	54	134	212
	[15,6]-d6e7d	$(2^3)2^6$	1,2,2,2,4,4	1	10	32	6	30	142	212
	[15,6]-d6e7e	$(2^3)2^4 3$	2,3,3,3,4	1	10	32	0	28	132	180
	[15,6]-d6e7f	$(2^3)2^4 3$	2,3,4,6	1	10	32	2	18	146	212
	[15,6]-e7⊕d8I	$(2^5)2^{10} 3 \cdot 7$	7,8	4	7	29	30	74	194	256
	[15,6]-e7⊕d8II	$(2^5)2^{10} 3^2 7$	7,8	4	7	29	14	138	98	32

	name	group order (sign change)perm.	orbit lengths	swe coefficients c^2	c^3	c^4	b^4	b^4c^1	b^4c^2	b^8
	[15,6]-e7⊕e8I	$(2^2)2^9 3^2 7$	7,8	0	7	21	30	90	138	240
	[15,6]-e7⊕e8II	$(2^2)2^9 3^2 7^2$	7,8	0	7	21	14	154	42	16
	[15,6]-e7d8a	$(2^5)2^8 3^2$	1,2,6,6	4	7	29	6	82	138	96
	[15,6]-e7d8b	$(2^5)2^8$	1,2,4,4,4	4	7	29	2	54	158	128
	[15,6]-e7d8c	$(2^5)2^8 3$	1,2,2,4,6	4	7	29	14	58	178	192
	[15,6]-e7d8d	$(2^5)2^8$	1,2,4,4,4	4	7	29	2	38	174	160
	[15,6]-e7d8e	$(2^5)2^7$	1,2,4,8	4	7	29	0	32	176	160
	[15,6]-e7d8f	$(2^5)2^7$	1,2,4,4,4	4	7	29	6	34	186	192
	[15,6]-e7d8g	$(2^5)2^7 3$	1,6,8	4	7	29	0	16	192	192
	[15,6]-e7d8h	$(2^5)2^{10} 3$	1,6,8	4	7	29	6	66	154	128
	[15,6]-e7d8i	$(2^5)2^{10} 3$	1,6,8	4	7	29	6	50	170	160
	[15,6]-e7d8j	$(2^5)2^9$	1,2,4,8	4	7	29	2	22	190	192
	[15,6]-e7d8k	$(2^5)2^9 3$	1,6,8	4	7	29	14	26	210	256
	[15,6]-e7d8l	$(2^5)2^9 3$	3,4,8	4	7	29	2	6	206	224
	[15,7]-d14a	$(2^2)2^6 7$	1,14	0	7	21	14	42	154	240
	[15,7]-d14b	$(2^2)2^6$	1,2,4,4,4	0	7	21	2	38	134	176
*	**[15,7]-d14c**	$(2^2)2^6$	1,2,4,4,4	0	7	21	0	32	136	176
	[15,7]-d14d	$(2^2)2^6$	1,2,4,4,4	0	7	21	6	34	146	208
	[15,7]-d14e	$(2^2)2^6 3$	1,4,4,6	0	7	21	6	50	130	176
*	**[15,7]-d14f**	$(2^2)2^6 3^2$	1,2,12	0	7	21	0	48	120	144
*	**[15,7]-d14g**	$(2^2)2^6$	1,2,4,4,4	0	7	21	0	16	152	208
	[15,7]-d14h	$(2^2)2^6 3$	1,2,12	0	7	21	6	18	162	240
	[15,7]-d14i	$(2^2)2^6 3$	1,4,4,6	0	7	21	2	22	150	208
	[15,7]-d14j	$(2^2)2^6 5$	1,4,10	0	7	21	2	22	150	208
	[15,7]-d14k	$(2^2)2^6$	1,2,4,4,4	0	7	21	2	22	150	208
*	**[15,7]-d14l**	$(2^2)2^6 3 \cdot 7$	1,14	0	7	21	0	0	168	240
	[15,7]-e7e8a	$(2^2)2^6 3$	1,2,2,4,6	0	7	21	2	54	118	144
	[15,7]-e7e8b	$(2^2)2^6 3^2$	1,1,1,6,6	0	7	21	14	58	138	208
*	**[15,7]-e7e8c**	$(2^2)2^5$	1,2,4,4,4	0	7	21	0	32	136	176
	[15,7]-e7e8d	$(2^2)2^5$	1,2,2,2,4,4	0	7	21	6	34	146	208
	[15,7]-e7e8e	$(2^2)2^3 3$	1,4,4,6	0	7	21	2	22	150	208
	[15,7]-e7e8f	$(2^2)2^9 3^2$	1,6,8	0	7	21	6	66	114	144
	[15,7]-e7e8g	$(2^2)2^9 3^2$	1,6,8	0	7	21	6	82	98	112
	[15,7]-e7e8h	$(2^2)2^6 3^2 7$	1,1,6,7	0	7	21	6	82	98	112
	[15,7]-e7e8i	$(2^2)2^8$	1,2,4,8	0	7	21	2	38	134	176
	[15,7]-e7e8j	$(2^2)2^8 3$	1,6,8	0	7	21	14	42	154	240
	[15,7]-e7e8k	$(2^2)2^6 3$	1,2,2,4,6	0	7	21	2	38	134	176
	[15,7]-e7e8l	$(2^2)2^8 3$	3,4,8	0	7	21	2	22	150	208
*	**[15,7]-e7e8m**	$(2^2)2^6 3$	3,4,8	0	7	21	0	16	152	208
	[15,7]-e7e8n	$(2^2)2^6 3$	3,4,8	0	7	21	6	18	162	240
*	**[15,7]-e7e8o**	$(2^2)2^5 3$	1,4,4,6	0	7	21	0	16	152	208
*	**[15,7]-e7e8p**	$(2^2)2^6 3 \cdot 7$	7,8	0	7	21	0	0	168	240

VI. Appendix II: Generator Matrices of the Indecomposable Self-Dual Codes of Lengths 10 & 11 and Codes Having Minimum Lee Weight 6

[10,2]-d4 =

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&1&2\\ 1&1&0&1&1&1&1&1&0&1\\ 2&2&0&0&0&0&0&0&0&0\\ 2&0&2&2&0&0&0&0&0&0\\ 0&0&0&2&2&0&0&0&0&0\\ 0&0&0&2&0&2&0&0&0&0\\ 0&0&0&2&0&0&2&0&0&0\\ 0&0&0&2&0&0&0&2&0&0 \end{bmatrix}$$

[10,3]-d4d6 =

$$\begin{bmatrix} 1&1&1&0&0&0&0&1&2&0\\ 0&0&0&1&1&1&0&2&1&0\\ 0&0&0&1&1&0&1&0&2&1\\ 2&2&0&0&0&0&0&0&0&0\\ 2&0&2&0&0&0&0&0&0&0\\ 0&0&0&2&2&0&0&0&0&0\\ 0&0&0&2&0&2&2&0&0&0 \end{bmatrix}$$

[10,4]-d10a =

$$\begin{bmatrix} 1&1&1&0&0&0&1&0&0&0\\ 1&1&0&1&0&0&2&1&0&0\\ 1&1&0&0&1&0&2&2&1&0\\ 1&1&0&0&0&1&2&2&2&1\\ 2&2&0&0&0&0&0&0&0&0\\ 2&0&2&2&2&2&0&0&0&0 \end{bmatrix}$$

[10,4]-d10b =

$$\begin{bmatrix} 1&1&1&0&0&0&1&0&0&0\\ 1&1&0&1&0&0&2&1&0&2\\ 1&1&0&0&1&0&2&2&1&0\\ 1&1&0&0&0&1&2&0&2&1\\ 2&2&0&0&0&0&0&0&0&0\\ 2&0&2&2&2&2&0&0&0&0 \end{bmatrix}$$

[11,3]-d6 =

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&1&0&0\\ 1&1&0&1&0&0&0&0&2&1&0\\ 1&0&1&1&1&1&1&1&2&2&1\\ 2&2&0&0&2&0&0&0&0&0&0\\ 2&0&2&2&2&0&0&0&0&0&0\\ 0&0&0&0&2&2&0&0&0&0&0\\ 0&0&0&0&2&0&2&0&0&0&0\\ 0&0&0&0&2&0&0&2&0&0&0 \end{bmatrix}$$

[11,4]-d4e7 =

$$\begin{bmatrix} 1&1&1&0&0&0&0&1&2&0&0\\ 0&0&0&1&0&1&1&2&1&0&0\\ 0&0&0&1&1&0&1&0&2&1&0\\ 0&0&0&1&1&1&0&0&2&2&1\\ 2&2&0&0&0&0&0&0&0&0&0\\ 2&0&2&0&0&0&0&0&0&0&0\\ 0&0&0&0&2&2&2&0&0&0&0 \end{bmatrix}$$

[14,6]-2e7c =

$$\begin{bmatrix} 0&1&1&1&0&0&0&0&1&0&0&0&0&2\\ 1&0&1&1&0&0&0&0&2&1&0&0&2&0\\ 1&1&0&1&0&0&0&0&2&2&1&2&0&0\\ 0&0&0&0&0&1&1&1&0&0&2&1&0&0\\ 0&0&0&0&1&0&1&1&0&2&0&2&1&0\\ 0&0&0&0&1&1&0&1&2&0&0&2&2&1\\ 2&2&2&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&2&2&2&0&0&0&0&0&0&0 \end{bmatrix}$$

[15,4]-f $=$

$$\begin{bmatrix} 1&1&1&1&1&1&1&0&0&0&0&1&0&0&0\\ 1&1&1&1&0&0&0&1&1&1&0&0&1&0&0\\ 1&1&0&0&1&1&0&1&1&0&1&0&0&1&0\\ 1&0&1&0&1&0&1&1&0&1&1&0&0&0&1\\ 2&2&0&0&0&0&0&2&2&0&0&0&0&0&0\\ 2&2&0&2&0&2&0&0&2&0&0&0&0&0&2\\ 2&0&0&0&0&2&0&0&2&0&2&0&0&0&0\\ 2&0&0&2&0&2&2&0&0&0&0&0&0&0&0\\ 2&2&0&0&2&2&0&0&0&0&0&0&0&0&0\\ 2&2&2&2&0&0&0&0&0&0&0&0&0&0&0\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

[15,5]-3d4a $=$

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&0&0&1&2&2&2&2\\ 0&0&0&1&1&1&0&0&0&0&2&1&0&2&2\\ 0&0&0&0&0&0&1&1&1&0&2&0&1&2&2\\ 1&1&0&1&1&0&1&1&0&1&0&0&0&1&0\\ 1&0&1&1&0&1&1&0&1&1&0&0&0&0&1\\ 0&2&0&0&2&0&0&0&0&0&2&2&0&0&0\\ 0&0&2&0&0&2&0&0&0&0&2&2&0&0&0\\ 0&2&0&0&0&0&0&2&0&0&2&0&2&0&0\\ 0&0&2&0&0&0&0&0&2&0&2&0&2&0&0\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

[15,5]-3d4d $=$

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&0&0&1&2&2&0&0\\ 0&0&0&1&1&1&0&0&0&0&2&1&2&0&0\\ 0&0&0&0&0&0&1&1&1&0&2&2&1&0&0\\ 1&1&0&1&1&0&1&1&0&1&2&2&2&1&0\\ 1&0&1&1&0&1&1&0&1&1&2&2&2&0&1\\ 0&2&0&0&2&0&0&0&0&0&2&2&0&0&0\\ 0&0&2&0&0&2&0&0&0&0&2&2&0&0&0\\ 0&2&0&0&0&0&0&2&0&0&2&0&2&0&0\\ 0&0&2&0&0&0&0&0&2&0&2&0&2&0&0\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

[15,6]-d6d8d $=$

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&0&1&2&2&0&2&0\\ 1&1&0&1&0&0&0&0&0&0&1&0&0&2&2\\ 0&0&0&0&1&1&1&0&0&2&0&1&0&0&2\\ 0&0&0&0&1&1&0&1&0&0&0&2&1&0&2\\ 0&0&0&0&1&1&0&0&1&2&2&2&2&1&0\\ 1&0&1&1&0&1&1&1&1&2&0&0&0&2&1\\ 2&2&0&0&0&0&2&0&0&0&0&2&0&0&0\\ 0&0&0&0&0&2&2&0&0&0&0&0&2&2&0\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

[15,6]-d6d8f $=$

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&0&1&0&2&0&2&0\\ 1&1&0&1&0&0&0&0&0&2&1&2&0&0&0\\ 0&0&0&0&1&1&1&0&0&2&2&1&0&2&0\\ 0&0&0&0&1&1&0&1&0&0&0&2&1&0&0\\ 0&0&0&0&1&1&0&0&3&2&0&0&2&1&0\\ 1&0&1&1&1&0&1&1&3&2&2&2&2&2&1\\ 2&2&0&0&0&0&2&0&0&0&0&2&0&0&0\\ 0&0&0&0&0&2&0&0&0&0&0&2&2&2&0\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

[15,6]-d6d8g $=$

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&0&1&0&2&2&0&0\\ 1&1&0&1&0&0&0&0&0&2&1&2&0&2&0\\ 0&0&0&0&1&1&1&0&0&2&2&1&0&0&0\\ 0&0&0&0&1&1&0&1&0&2&0&2&1&0&0\\ 0&0&0&0&1&1&0&0&1&0&2&2&2&1&0\\ 1&0&1&1&1&0&1&1&1&2&2&2&2&2&1\\ 2&2&0&0&0&0&0&0&2&0&0&0&0&2&0\\ 2&0&0&0&0&0&0&0&0&2&2&0&0&0&2\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

[15,6]-d6d8l $=$

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&0&1&0&2&2&0&0\\ 1&1&0&1&0&0&0&0&0&2&1&2&0&2&0\\ 0&0&0&0&1&1&1&0&0&2&2&1&0&2&0\\ 0&0&0&0&1&1&0&1&0&2&0&2&1&0&0\\ 0&0&0&0&1&1&0&0&1&0&2&0&2&1&0\\ 1&0&1&1&1&0&1&1&1&2&2&2&2&2&1\\ 2&2&0&0&0&0&2&0&0&0&0&2&0&0&0\\ 0&0&0&0&2&0&2&0&0&0&0&0&2&2&0\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

[15,7]-d14c =

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&1&0&0&0&2&0&0\\ 1&1&0&1&0&0&0&0&2&1&0&0&0&2&0\\ 1&1&0&0&1&0&0&0&2&2&1&0&0&0&0\\ 1&1&0&0&0&1&0&0&2&2&2&1&0&2&0\\ 1&1&0&0&0&0&1&0&0&2&2&2&1&0&0\\ 1&1&0&0&0&0&0&1&2&0&2&0&2&1&0\\ 1&0&1&1&1&1&1&1&2&2&2&2&2&2&1\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

[15,7]-d14f =

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&1&0&0&0&0&2&0\\ 1&1&0&1&0&0&0&0&2&1&0&2&0&0&0\\ 1&1&0&0&1&0&0&0&2&2&1&0&0&0&0\\ 1&1&0&0&0&1&0&0&2&0&2&1&0&0&0\\ 1&1&0&0&0&0&1&0&2&2&2&2&1&0&0\\ 1&1&0&0&0&0&0&1&0&2&2&2&2&1&0\\ 1&0&1&1&1&1&1&1&2&2&2&2&2&2&1\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

[15,7]-d14g =

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&1&0&0&0&2&2&0\\ 1&1&0&1&0&0&0&0&2&1&0&0&2&0&0\\ 1&1&0&0&1&0&0&0&2&2&1&0&0&2&0\\ 1&1&0&0&0&1&0&0&2&2&2&1&0&0&0\\ 1&1&0&0&0&0&1&0&0&0&2&2&1&0&0\\ 1&1&0&0&0&0&0&1&0&2&0&2&2&1&0\\ 1&0&1&1&1&1&1&1&2&2&2&2&2&2&1\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

[15,7]-d14l =

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&1&0&0&2&0&2&0\\ 1&1&0&1&0&0&0&0&2&1&0&0&2&0&0\\ 1&1&0&0&1&0&0&0&2&2&1&0&0&2&0\\ 1&1&0&0&0&1&0&0&0&2&2&1&0&0&0\\ 1&1&0&0&0&0&1&0&2&0&2&2&1&0&0\\ 1&1&0&0&0&0&0&1&0&2&0&2&2&1&0\\ 1&0&1&1&1&1&1&1&2&2&2&2&2&2&1\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

[15,7]-e7e8c =

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&1&0&0&2&0&0&0\\ 1&1&0&1&0&0&0&0&2&1&0&0&2&0&0\\ 1&0&1&1&0&0&0&0&2&2&1&0&0&2&0\\ 0&0&0&0&1&1&1&0&2&0&0&1&0&0&0\\ 0&0&0&0&1&1&0&1&0&2&0&2&1&0&0\\ 0&0&0&0&1&0&1&1&0&0&2&2&2&1&0\\ 0&0&0&0&0&1&1&1&0&0&0&2&2&2&1\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

[15,7]-e7e8m =

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&1&0&0&0&2&0&2\\ 1&1&0&1&0&0&0&0&2&1&0&2&2&0&0\\ 1&0&1&1&0&0&0&0&2&2&1&2&0&2&0\\ 0&0&0&0&1&1&1&0&0&2&2&1&0&0&0\\ 0&0&0&0&1&1&0&1&2&2&0&2&1&0&2\\ 0&0&0&0&1&0&1&1&0&0&2&2&2&1&0\\ 0&0&0&0&0&1&1&1&2&0&0&2&0&2&1\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

[15,7]-e7e8o =

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&1&0&0&2&0&0&0\\ 1&1&0&1&0&0&0&0&2&1&0&0&0&2&0\\ 1&0&1&1&0&0&0&0&2&2&1&0&0&0&2\\ 0&0&0&0&1&1&1&0&2&0&0&1&0&2&0\\ 0&0&0&0&1&1&0&1&0&0&0&2&1&0&0\\ 0&0&0&0&1&0&1&1&0&2&0&0&2&1&0\\ 0&0&0&0&0&1&1&1&0&0&2&2&2&2&1\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

[15,7]-e7e8p =

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&1&0&0&2&0&0&2\\ 1&1&0&1&0&0&0&0&2&1&0&0&2&0&2\\ 1&0&1&1&0&0&0&0&2&2&1&0&0&2&2\\ 0&0&0&0&1&1&1&0&2&0&0&1&0&2&0\\ 0&0&0&0&1&1&0&1&0&2&0&2&1&0&0\\ 0&0&0&0&1&0&1&1&0&0&2&0&2&1&0\\ 0&0&0&0&0&1&1&1&2&2&2&2&2&2&1\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

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