Differential Equations with a Fractal Character

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A problem from mechanism design: finding the optimal motion profile for a transfer-dwell-return-dwell cam, has lead to the discovery of a curious family of functions. Functions that are self-similar to one of their derivatives will be said to satisfy a fractal differential equation. We will present several examples and consider this class of functions in the context of Taylor and Fourier series approximations.

Joint work with Leon Q. Brin.

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Automation

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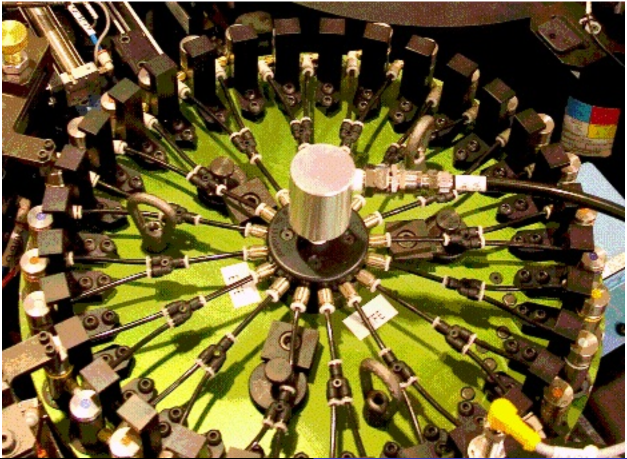
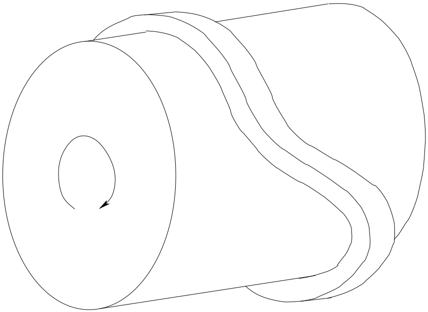
- There are linear and circular transfer systems

a linear turnkey system

a rotary system



barrel cam



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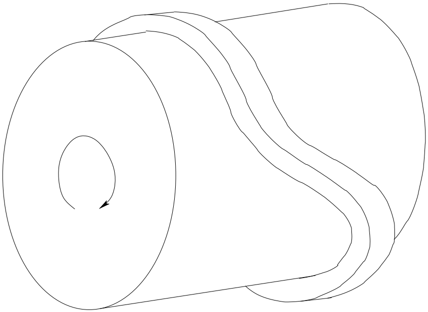
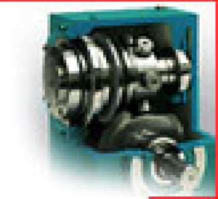
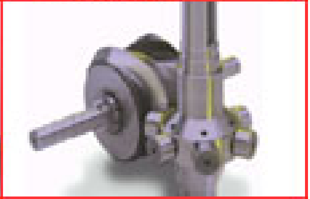


Cam followers are pairs of rollers that are often pre-stressed against the cam rib.

roller gear





the venerable geneva mechanism

http://www.youtube.com/watch?v=mEShmrrdFQw

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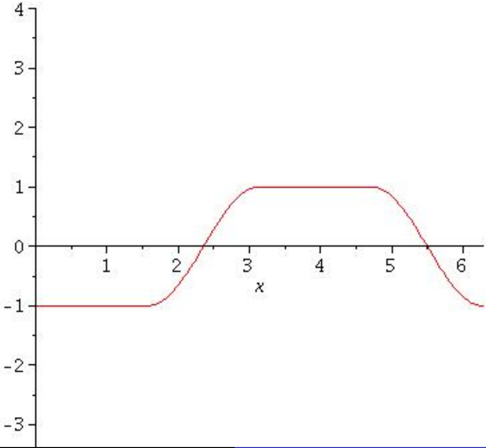
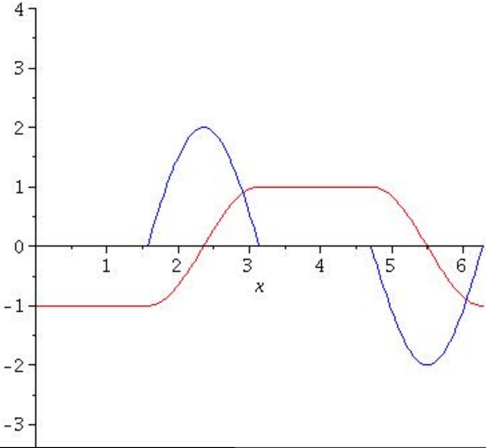
- The two dwells occur at -1 and 1.

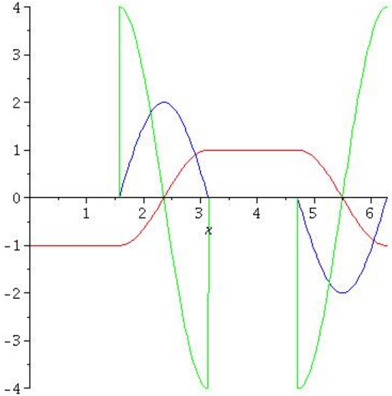
Why not just “patch in” a sine curve?



velocity of piecewise curve using sine



acceleration of piecewise curve using sine



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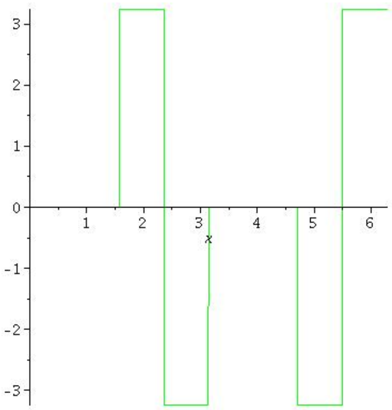
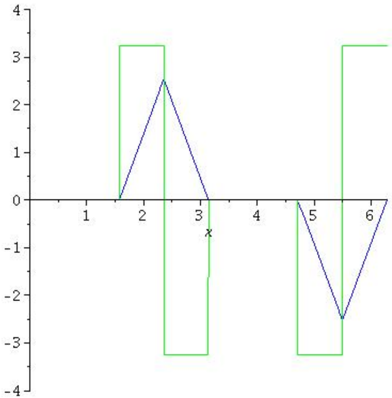
- The motion profile will be piecewise constants and quadratics.

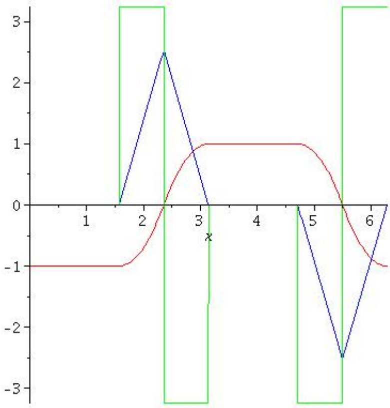
A constant acceleration scheme



A constant acceleration scheme (acc. and vel.)



A constant acceleration scheme (pos., vel. & acc.)



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- Infinitudes in the jerk produce very high transient accelerations. (shocks)

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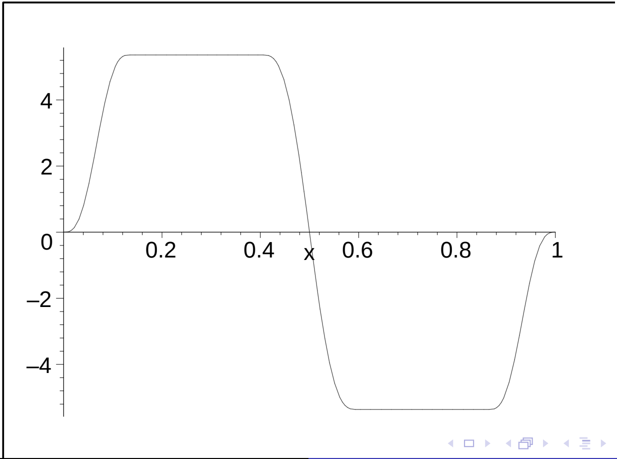
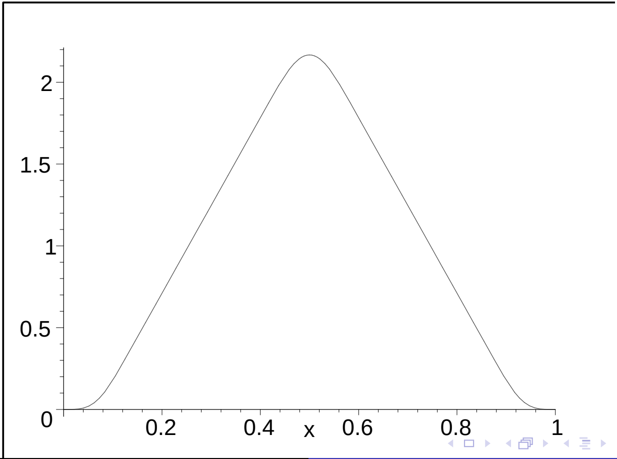
- The acceleration profile needs to accomplish exactly the same task as the position – transfer from one fixed value to another smoothly.
- The second derivative should be cobbled together out of constants and pieces that look like scaled versions of the original function.

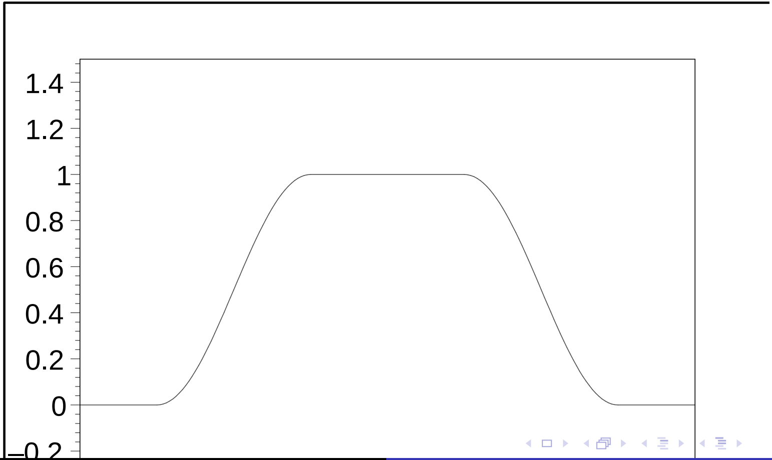
the acceleration profile of a the "fractal cam"



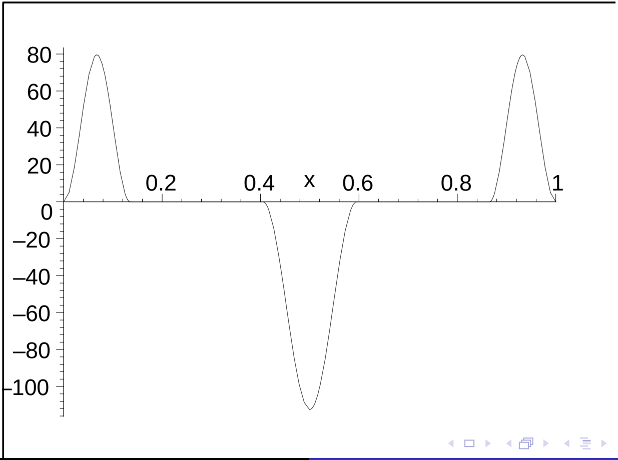
the velocity profile of a the "fractal cam"



the position profile of a the “fractal cam”



the jerk profile of a the "fractal cam"



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- If this iteration approaches a limit – you've found the solution to your fractal diffEQ.

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- http://ifs-tools.sourceforge.net/

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- "Most" Taylor expansions around a point converge on miniscule intervals.

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- Outside the dwell intervals this Taylor series clearly has no predictive power.

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- Okay, so Taylor wasn't the right way to go, but Fourier . . .

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- The successive Fourier approximations don't converge terribly well to the cam.

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- Are there other physically meaningful fractal diffEQs?
- How contractive is the IFS for a given fractal diffEQ?