Differential Equations with a Fractal Character

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A problem from mechanism design: finding the optimal motion profile for a transfer-dwell-return-dwell cam, has lead to the discovery of a curious family of functions. Functions that are self-similar to one of their derivatives will be said to satisfy a fractal differential equation. We will present several examples and consider this class of functions in the context of Taylor and Fourier series approximations.
Joint work with Leon Q. Brin.
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Automation

- Parts are moved from one station to another where they are incrementally worked on.
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- There are linear and circular transfer systems
a linear turnkey system
a rotary system
barrel cam
Cam followers are pairs of rollers that are often pre-stressed against the cam rib.
roller gear
the venerable geneva mechanism

http://www.youtube.com/watch?v=mEShmrrdFQw
Conventions

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- Motion consists of a cycle of transfer-dwell-return-dwell.
- Use $90^\circ$ for each of the four phases.
- The two dwells occur at $-1$ and $1$. 
Why not just “patch in” a sine curve?
velocity of piecewise curve using sine
acceleration of piecewise curve using sine
problems

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- Infinite jerk.
minimizing the maximum acceleration

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Apply the full acceleration until we’re halfway there then apply full deceleration until we arrive.
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- Apply the full acceleration until we’re halfway there then apply full deceleration until we arrive.
- The motion profile will be piecewise constants and quadratics.
A constant acceleration scheme
A constant acceleration scheme (acc. and vel.)
A constant acceleration scheme (pos., vel. & acc.)
problems with constant acceleration scheme

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- Mechanisms are sensitive to the FORCES acting on their components.
- Forces are proportional to acceleration. \( (F=ma) \)
- Jerk can’t be sensed can it?
- Infinitudes in the jerk produce very high transient accelerations. (shocks)
What should we do to be kind to the jerk?

- Self-similarity (in the 2nd derivative)
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- Self-similarity (in the 2nd derivative)
- The acceleration profile needs to accomplish exactly the same task as the position – transfer from one fixed value to another smoothly.
- The second derivative should be cobbled together out of constants and pieces that look like scaled versions of the original function.
the acceleration profile of a the “fractal cam”
the velocity profile of a the "fractal cam"
the position profile of a the “fractal cam”
the jerk profile of a the "fractal cam"
Iterated solution of fractal diffEQ

- Start with an arbitrary function $f$ (e.g. linear) as the transfer curve.
Iterated solution of fractal diffEQ

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▶ Iterate over the following:
Iterated solution of fractal diffEQ

- Start with an arbitrary function $f$ (e.g. linear) as the transfer curve.
- Iterate over the following:
  - Write a new acceleration profile with (piecewise) scaled copies of $f$ and constants.
Iterated solution of fractal diffEQ

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  - Integrate twice to find a new $f$. 

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Fractal DiffEQ
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  - Constants and scaling have to be chosen so that the new $f$ has range $[-1, 1]$. 
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  - Constants and scaling have to be chosen so that the new \( f \) has range \([-1, 1]\).
- If this iteration approaches a limit – you’ve found the solution to your fractal diffEQ.
The process just described can be thought of as an iterated function system (IFS).
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http://ifs-tools.sourceforge.net/
Taylor series of a fractal cam

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- The Taylor series at any point is just a polynomial – it converges to the fractal cam profile only in a small region.
- “Most” Taylor expansions around a point converge on miniscule intervals.
Taylor series aren’t particularly suitable.

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- Consider the intervals where the profile “dwell}s.”
- There, the Taylor expansion is identically constant.
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- Consider the intervals where the profile “dwell.
- There, the Taylor expansion is identically constant.
- Outside the dwell intervals this Taylor series clearly has no predictive power.
Clearly the fractal cam profile is periodic.
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Taylor series for functions like sin and cos converge pretty slowly.
Clearly the fractal cam profile is periodic.

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Okay, so Taylor wasn’t the right way to go, but Fourier ...
We can approximate the fractal cam profile to arbitrary precision numerically.
Fourier series

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- Fourier coefficients are also easy to produce numerically.
- The coefficients show no discernable pattern.
- The successive Fourier approximations don’t converge terribly well to the cam.
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Are there other physically meaningful fractal diffEQs?

How contractive is the IFS for a given fractal diffEQ?
questions

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▶ alternate trigonometries from “fat circles” (e.g. $x^4 + y^4 = 1$)
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- Are there other bases for the space of periodic functions, relative to which convergence will be faster?
- Alternate trigonometries from “fat circles” (e.g. \(x^4 + y^4 = 1\))
- Are there other physically meaningful fractal diffEQs?
- How contractive is the IFS for a given fractal diffEQ?